Contributions to the extension theory in Hilbert spaces

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Chapter 5

A factorization approach to the extension theory of nonnegative relations

The Kreĭn-von Neumann and the Friedrichs extensions of a nonnegative linear operator or relation are characterized in terms of factorizations. These factorizations lead to a novel approach to the transversality and equality of the Kreĭn-von Neumann and the Friedrichs extensions and to the notion of positive closability (the Kreĭn-von Neumann extension being an operator). Furthermore, all extremal extensions of the nonnegative operator or relation are characterized in terms of analogous factorizations. This approach for the general case of nonnegative linear relations in a Hilbert space extends the applicability of such factorizations. In fact, the extension theory of densely and nondensely defined nonnegative relations or operators fits in the same framework. In particular, all extremal extensions of a bounded nonnegative operator are characterized.

5.1 Introduction

To illustrate the factorizations introduced in this chapter consider the following simple completion problem. Let $\mathfrak{H}$ be a Hilbert space with the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let S_{11} be a nonnegative bounded linear operator in $\mathfrak{H}_1$, let S_{21} be a bounded linear operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, and let $S_{12} = S_{21}^*$. The usual form of the completion problem requires to determine all bounded linear operators S_{22} in $\mathfrak{H}_2$, such that

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & * \end{pmatrix} \tag{5.1.1}$$

becomes a nonnegative bounded linear operator in $\mathfrak{H}$, cf. [14], [32], [37], [55], [56], [60], [62]. This completion problem has a solution if and only if $\operatorname{ran} S_{21}^* \subset \operatorname{ran} S_{11}^{1/2}$. To put this completion problem in a more general framework introduce the following linear relation

$$V = \{ \{h, k\} \in \mathfrak{H}_2 \times \overline{\operatorname{ran}}\, S_{11} : S_{11}^{1/2} k = S_{21}^* h \}, \tag{5.1.2}$$

as a subset of the Cartesian product $\mathfrak{H}_2 \times \overline{\operatorname{ran}}\, S_{11}$. Clearly, V is linear and closed. Furthermore, if $\{0, k\} \in V$ then by definition $k \in \ker S_{11} \cap \overline{\operatorname{ran}}\, S_{11} = \{0\}$, which means that V is the graph of a closed linear operator. The closed linear operator V gives rise to the following "solution" of (5.1.1):

$$\begin{pmatrix} S_{11}^{1/2} \\ V^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & V \end{pmatrix}, \tag{5.1.3}$$

where the product is in the sense of linear relations. If $\operatorname{dom} V = \mathfrak{H}_2$ (i.e., $\operatorname{ran} S_{21}^* \subset \operatorname{ran} S_{11}^{1/2}$), then $S_{11}^{1/2} V = S_{12}$, $S_{22} = V^*V$, and (5.1.3) gives the smallest solution of (5.1.1). If V is densely defined, then (5.1.3) still gives the smallest solution of (5.1.1), but now $S_{22} = V^*V$ is in general an unbounded operator. Moreover, if V is not densely defined then $S_{22} = V^*V$ is a nonnegative relation (i.e., a multivalued operator), and still (5.1.3) provides a smallest solution in the sense of relations, cf. [24], [37]. Furthermore, in the sense of relations, the completion problem (5.1.1) has always the following nonnegative solution:

$$\begin{pmatrix} S_{11}^{1/2} \\ O^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & O \end{pmatrix}, \tag{5.1.4}$$

where O is the trivial linear relation from $\mathfrak{H}_2$ to $\overline{\operatorname{ran}}\, S_{11}$ and its adjoint O^* is given by $O^* = \overline{\operatorname{ran}}\, S_{11} \times \mathfrak{H}_2$. In fact, in the sense of relations, (5.1.4) is the largest solution of (5.1.1). All the other solutions of (5.1.1) are between these extreme solutions. In particular, if R is an arbitrary restriction of the closed operator V, then

$$\begin{pmatrix} S_{11}^{1/2} \\ R^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & R^{**} \end{pmatrix}, \tag{5.1.5}$$

is between (5.1.3) and (5.1.4), and hence a solution to the completion problem (5.1.1). For a proper interpretation of solutions to (5.1.1), introduce the operator S by

$$S = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}. \tag{5.1.6}$$

Clearly, S is a nonnegative bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}$. The completion problem (5.1.1) can now be interpreted as an extension problem for S: the nonnegative solutions to (5.1.3) correspond to the nonnegative selfadjoint relation extensions of S. The factorizations of the extreme solutions of (5.1.1), i.e., of the extreme extensions of S in (5.1.6), persist in the general extension theory of nonnegative relations. It is the purpose of this chapter to develop this approach of factorizations for the general case of the extension theory of nonnegative operators and relations and to study the consequences.

Let S be any nonnegative linear relation in a Hilbert space $\mathfrak{H}$. Then there are two nonnegative selfadjoint extensions of S in $\mathfrak{H}$, namely the Kreĭn-von Neumann extension S_N and the Friedrichs extension S_F, which are extreme in the sense that all other nonnegative selfadjoint extensions of S lie between them: if H is a nonnegative selfadjoint extension of S, then

$$S_N \le H \le S_F, \tag{5.1.7}$$

where the inequalities are in the sense of the corresponding resolvent operators:

$$(S_F + a)^{-1} \leq (H + a)^{-1} \leq (S_N + a)^{-1}, \quad a > 0, \tag{5.1.8}$$

or, equivalently, in the sense of the corresponding closed nonnegative forms:

$$\mathfrak{t}_{S_N} \leq \mathfrak{t}_H \leq \mathfrak{t}_{S_F}. \tag{5.1.9}$$

When S is the nonnegative operator in (5.1.6) associated with the completion problem (5.1.1), then the Kreĭn-von Neumann extension S_N is given by (5.1.3) and the Friedrichs extension S_F is given by (5.1.4). The general theory of nonnegative selfadjoint extensions of densely defined nonnegative operators is due to M.G. Kreĭn [41], cf. also [65], [68], [69]. T. Ando and K. Nishio [2] have considered operator extensions in the case when S is a not necessarily densely defined operator. The general case involving relations goes back to [20], see also Chapter 4 for the interpretation of (5.1.9) in this case. For a review of Kreĭn's work (in the context of relations), see [37]. It will be shown in the present chapter that the Kreĭn-von Neumann extension S_N in (5.1.7) has a factorization $S_N = J^{**}J^*$, where J is a linear relation from an auxiliary space $\mathfrak{H}_S$ to $\mathfrak{H}$, and that the Friedrichs extension in (5.1.7) has a similar factorization. Such factorizations go back to J. Stochel, Z. Sebestyén, and coworkers (see [50], [49], [57], [59], [58]) for the case that S is a densely defined operator or under a condition which guarantees that the Kreĭn-von Neumann extension S_N is an operator. The factorizations of S_N and S_F in the general case provide a novel approach to notions such as disjointness, transversality, and equality of S_N and S_F; and to the notion of positive closability of S (S_N being an operator). A nonnegative selfadjoint extension H of S is called extremal when

$$\inf\{ (f' - h', f - h) : \{h, h'\} \in S \} = 0 \quad \text{for all} \quad \{f, f'\} \in H. \tag{5.1.10}$$

This definition goes back at least to Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ [11]. In the densely defined case the factorization of the extremal extensions was studied in [10]. In the present chapter the extremal extensions are characterized by factorizations in the general case. In particular, the extremal extensions of S in (5.1.6) are precisely the solutions of (5.1.1) given by (5.1.5).

The contents of this chapter are now briefly listed. In Section 5.2 some results concerning the Kreĭn-von Neumann and the Friedrichs extensions are recalled. Furthermore this section provides a simple treatment of the disjointness and transversality for nonnegative selfadjoint extensions of a nonnegative relation. The generalization of the construction of the Kreĭn-von Neumann and the Friedrichs extensions in the sense of Sebestyén and Stochel can be found in Section 5.3. Disjointness, transversality and equality of the Kreĭn-von Neumann and Friedrichs extensions from the point of view of factorizations are discussed in Section 5.4. Also the question whether the Kreĭn-von Neumann extension is a (bounded) operator is discussed in this section. The treatment of extremal extensions and their factorizations can be found in Section 5.5. The case of a nonnegative bounded operator and its extremal extensions is treated in Section 5.6. This provides the link with the completion problem discussed above. In Section 5.7 the extremal extensions will be made explicit for some simple nonnegative relations .

5.2 The extreme extensions

This section contains some general information concerning nonnegative selfadjoint extensions of a nonnegative relation S. In particular, the Kreĭn-von Neumann and the Friedrichs extensions are introduced. Furthermore there is a simple treatment of the transversality of any nonnegative selfadjoint extension of S and its Friedrichs extension.

5.2.1 Some general remarks

Let S be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$ and define on $\operatorname{dom}\mathfrak{t} = \operatorname{dom} S$

$$\mathfrak{t}[f,g] = (f',g), \quad \{f,f'\},\{g,g'\} \in S. \tag{5.2.1}$$

Then (5.2.1) gives rise to a nonnegative form since

$$(f',g) = (f,g') = (f'',g) \geq 0, \quad \{f,f'\},\{f,f''\},\{g,g'\} \in S.$$

In fact, the form $\mathfrak{t}$ in (5.2.1) is closable, cf. [40]. The closure $\bar{\mathfrak{t}}$ of the form $\mathfrak{t}$ in (5.2.1) is nonnegative and induces a nonnegative selfadjoint relation S_F which is the orthogonal sum of the selfadjoint operator induced by the form $\bar{\mathfrak{t}}$ in $\overline{\operatorname{dom}} S$ (cf. [40]) and the multivalued part $\{0\} \times \operatorname{mul} S^*$ (cf. [24] and Chapter 4). The nonnegative selfadjoint relation S_F is an extension of S and has the same lower bound as S, cf. [19]. By construction $\operatorname{mul} S_F = \operatorname{mul} S^*$, so that the Friedrichs extension is an operator if and only if S is densely defined (and necessarily an operator). For a nonnegative relation S introduce the space $\operatorname{dom}[S]$ as the set of all $f \in \mathfrak{H}$ for which there exists a sequence $(\{f_n, f_n'\}) \subset S$ such that

$$f_n \to f, \quad (f_n' - f_m', f_n - f_m) \to 0 \quad m,n \to \infty. \tag{5.2.2}$$

It can be shown that $\operatorname{dom}[S] = \operatorname{dom}[S_F] = \operatorname{dom} S_F^{1/2}$, and that

$$S_F = \{\, \{f,f'\} \in S^* : f \in \operatorname{dom}[S] \,\}. \tag{5.2.3}$$

Moreover, the Friedrichs extension is the only selfadjoint extension of S whose domain is contained in $\operatorname{dom}[S]$.

If the relation S is nonnegative (selfadjoint), then likewise the formal inverse S^{-1} of S is nonnegative (selfadjoint). Hence the selfadjoint relation

$$S_N = ((S^{-1})_F)^{-1} \tag{5.2.4}$$

is also a nonnegative selfadjoint extension of S; in fact it is the Kreĭn-von Neumann extension of S, cf. [41], [2], [20]. In particular, S_N is the only selfadjoint extension of S whose range is contained in $\operatorname{ran}[S] := \operatorname{dom}[S^{-1}]$ and the following description holds

$$S_N = \{\, \{f,f'\} \in S^* : f' \in \operatorname{ran}[S] \,\}.$$

Notice also that $\ker S_N = \ker S^*$, and that $f' \in \operatorname{ran}[S]$ if and only if there exists a sequence $(\{f_n, f'_n\}) \subset S$, such that

$$f'_n \to f', \quad (f'_n - f'_m, f_n - f_m) \to 0, \quad m, n \to \infty. \tag{5.2.5}$$

The Kreĭn-von Neumann and the Friedrichs extensions are extreme nonnegative selfadjoint extensions of S: if H is any nonnegative selfadjoint extension of S, then (5.1.7) holds, where the inequalities are in the sense of resolvents (see (5.1.8)) or, equivalently, in the sense of the corresponding forms (see (5.1.9)), cf. Chapter 4. The following proposition is just a reformulation of the definition of the form domain $\operatorname{dom}[S]$ and of the form range $\operatorname{ran}[S]$, respectively, used in the construction of S_F and S_N.

Proposition 5.2.1. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let $\{f, f'\} \in S^*$. Then:*

(i) $\{f, f'\} \in S_F$ *if and only if*

$$\inf \left\{ \|f - h\|^2 + (f' - h', f - h) \,:\, \{h, h'\} \in S \right\} = 0;$$

(ii) $\{f, f'\} \in S_N$ *if and only if*

$$\inf \left\{ \|f' - h'\|^2 + (f' - h', f - h) \,:\, \{h, h'\} \in S \right\} = 0.$$

Sometimes the Kreĭn-von Neumann and the Friedrichs extensions can be given explicitly. If S is a nonnegative relation, then

$$S_N = S \,\widehat{+}\, (\ker S^* \times \{0\}), \tag{5.2.6}$$

if and only if

$$\operatorname{ran} S = \overline{\operatorname{ran}}\, S \cap \operatorname{ran} S^*, \tag{5.2.7}$$

and likewise, in view of (5.2.4),

$$S_F = S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*), \tag{5.2.8}$$

if and only if

$$\operatorname{dom} S = \overline{\operatorname{dom}}\, S \cap \operatorname{dom} S^*, \tag{5.2.9}$$

see [20], [35]. Observe that (5.2.7) is satisfied when $\operatorname{ran} S$ is closed, in which case (5.2.6) is valid. In particular, if S is closed and $m(S) > 0$, or if S^{-1} is a closed bounded operator, then (5.2.6) holds. Hence, if S is closed, then either $m(S_N) = 0$ or S is selfadjoint, in which case $S_N = S = S_F$, cf. [20].

A review of the above facts can be found in [37]; some further facts concerning inequalities between nonnegative selfadjoint relations and the corresponding forms can be found in Chapter 4, see also [13].

5.2.2 Disjointness and transversality

Assume that S is a symmetric relation in a Hilbert space $\mathfrak{H}$. The closed extensions A and B of S are said to be *disjoint* (with respect to S) if

$$S = A \cap B, \tag{5.2.10}$$

or equivalently, if S is closed and

$$S^* = \operatorname{clos}(A^* \mathbin{\widehat{+}} B^*). \tag{5.2.11}$$

Furthermore, if A and B are intermediate extensions of S, i.e. $S \subset A \subset S^*$ and $S \subset B \subset S^*$, then A and B are said to be *transversal* (with respect to S) if

$$S = A \cap B \text{ and } S^* = A \mathbin{\widehat{+}} B, \tag{5.2.12}$$

cf. [24]. It is clear that disjointness and transversality of A and B can be characterized by the conditions (2.3.6) and (2.3.7) in Lemma 2.3.2. For this purpose the following corollary is often sufficient.

Corollary 5.2.2. *Let S be a symmetric relation in a Hilbert space $\mathfrak{H}$ and let A and B be selfadjoint extensions of S. Then:*

(i) *A and B are disjoint if and only if for some (equivalently for every) $\lambda \in \rho(A) \cap \rho(B)$,*

$$\operatorname{ran}(S - \lambda) = \ker\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right); \tag{5.2.13}$$

(ii) *A and B are transversal if and only if for some (equivalently for every) $\lambda \in \rho(A) \cap \rho(B)$,*

$$\ker(S^* - \lambda) = \operatorname{ran}\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right). \tag{5.2.14}$$

Observe that if S is a symmetric relation in $\mathfrak{H}$, then $H := S \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_\lambda(S^*)$ is a restriction of S^* and, moreover, if S is closed,

$$\bar{\lambda} \in \rho(H), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Therefore, Lemma 2.3.1 shows that

$$S^* = S \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_\lambda(S^*) \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_{\bar{\lambda}}(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.2.15}$$

which is *von Neumann's decomposition* for the adjoint of a closed symmetric relation.

Originally, equivalent descriptions for disjointness and transversality have been established by means of boundary triplets using parameters in Kreĭn's formula, cf. [42] and [24]; for the case of sectorial linear relations, see also [6]. Disjointness and transversality play a role in the construction of boundary triplets, and more generally, of boundary relations, cf. [23].

Next, a concise treatment is given for the disjointness and the transversality of some nonnegative selfadjoint extension and the Friedrichs extension. The following observation is very useful.

Lemma 5.2.3. *Let A and B be nonnegative selfadjoint relations in a Hilbert space $\mathfrak{H}$ and let $a > 0$. If $A \leq B$, then*

$$\operatorname{dom} A^{1/2} = \operatorname{ran}\,\left((A+a)^{-1} - (B+a)^{-1}\right)^{1/2} + \operatorname{dom} B^{1/2}. \tag{5.2.16}$$

Proof. Since $a > 0$ it follows that $-a$ is in the resolvent sets of A and B. Hence $R(a) = (A+a)^{-1} - (B+a)^{-1} \in [\mathfrak{H}]$. It follows from the inequality $A \leq B$ that $R(a)$ is nonnegative, cf. [20] and Chapter 4. Furthermore,

$$(A+a)^{-1} = R(a) + (B+a)^{-1} = \begin{pmatrix} R(a)^{1/2} & (B+a)^{-1/2} \end{pmatrix} \begin{pmatrix} R(a)^{1/2} \\ (B+a)^{-1/2} \end{pmatrix},$$

which leads to

$$\operatorname{ran}\,(A+a)^{-1/2} = \operatorname{ran} R(a)^{1/2} + \operatorname{ran}\,(B+a)^{-1/2},$$

cf. [31]. The last result coincides with the decomposition (5.2.16). ■

Recall that if R is a nonnegative bounded linear operator in a Hilbert space $\mathfrak{H}$, then $\overline{\operatorname{ran}}\, R = \overline{\operatorname{ran}}\, R^{1/2}$, and $\operatorname{ran} R$ is closed if and only if $\operatorname{ran} R^{1/2}$ is closed.

Proposition 5.2.4. *Let S be a nonnegative linear relation and let H be a nonnegative selfadjoint extension of S. Then, for every $a > 0$,*

$$\operatorname{ran}\,\left((H+a)^{-1} - (S_F+a)^{-1}\right)^{1/2} = \ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}, \tag{5.2.17}$$

and, furthermore,

$$\operatorname{dom} H^{1/2} = \left(\ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}\right) + \operatorname{dom} S_F^{1/2}, \quad \textit{direct sum.} \tag{5.2.18}$$

Proof. Recall that $H \leq S_F$ via Kreĭn's inequalities. Hence, Lemma 5.2.3 may be applied with $A = H$ and $B = S_F$. Define $R(a) = (H+a)^{-1} - (S_F+a)^{-1}$, so that clearly $\operatorname{ran} R(a) \subset \ker\,(S^*+a)$. It suffices to show that

$$\operatorname{ran} R(a)^{1/2} = \ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}.$$

Clearly, it follows from (5.2.16) that $\operatorname{ran} R(a)^{1/2} \subset \operatorname{dom} H^{1/2}$, since $\operatorname{dom} S_F^{1/2} \subset \operatorname{dom} H^{1/2}$, cf. Chapter 4. Since $\operatorname{ran} R(a) \subset \ker\,(S^*+a)$, it follows that $\operatorname{ran} R(a)^{1/2} \subset \ker\,(S^*+a)$. Hence, the lefthand side is contained in the righthand side. Next the reverse inclusion will be shown. Let $f \in \ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}$, then $f \in \operatorname{dom} H^{1/2}$ implies that

$$f = h + k, \quad h \in \operatorname{dom} S_F^{1/2}, \quad k \in \operatorname{ran} R(a)^{1/2},$$

cf. (5.2.16). Since $k \in \ker\,(S^*+a)$, it follows that $h \in \ker\,(S^*+a) \cap \operatorname{dom} S_F^{1/2}$. Hence $h \in \ker\,(S_F+a)$ by (5.2.3), so that $h = 0$. Therefore $f \in \operatorname{ran} R(a)^{1/2}$, which completes the proof of (5.2.17).

The identity (5.2.18) follows now from Lemma 5.2.3. The sum is direct since (5.2.3) implies $\ker\,(S^*+a) \cap \operatorname{dom} S_F^{1/2} = \ker\,(S_F+a)$ and, hence, this set is trivial. ■

Proposition 5.2.5. *Let S be a nonnegative linear relation and let H be a nonnegative selfadjoint extension of S. Then H and S_F are transversal if and only if*

$$\ker\,(S^*+a) \subset \operatorname{dom} H^{1/2}, \quad a>0. \tag{5.2.19}$$

Furthermore, $H=S_F$ if and only if

$$\ker\,(S^*+a) \cap \operatorname{dom} H^{1/2} = \{0\}, \quad a>0. \tag{5.2.20}$$

Proof. Assume that (5.2.19) is satisfied. Then it follows from (5.2.17) that

$$\operatorname{ran}\,\left((H+a)^{-1}-(S_F+a)^{-1}\right)^{1/2} = \ker\,(S^*+a). \tag{5.2.21}$$

But then also

$$\operatorname{ran}\,\left((H+a)^{-1}-(S_F+a)^{-1}\right) = \ker\,(S^*+a).$$

Hence by Corollary 5.2.2 H and S_F are transversal. The converse statement is obtained by retracing these steps. This proves the first assertion of the proposition.

Observe that $H=S_F$ if and only if $\operatorname{dom} H^{1/2} = \operatorname{dom} S_F^{1/2}$, since S_F is the only selfadjoint extension of S whose domain is contained in $\operatorname{dom}[S_F] = \operatorname{dom} S_F^{1/2}$. It follows from (5.2.18) that $\operatorname{dom} H^{1/2} = \operatorname{dom} S_F^{1/2}$ if and only if (5.2.20) holds. This proves the last assertion of the proposition. ∎

Corollary 5.2.6. *Let S be a nonnegative linear relation. Then S_N and S_F are transversal if and only if*

$$\operatorname{dom} S^* \subset \operatorname{dom} S_N^{1/2}. \tag{5.2.22}$$

Proof. Clearly, if (5.2.22) holds then S_F and S_N are transversal, by Proposition 5.2.5.

Conversely, if S_N and S_F are transversal, then $\ker\,(S^*+a) \subset \operatorname{dom} S_N^{1/2}$ by Proposition 5.2.5. If $a>0$, then $-a \in \rho(S_N)$, and hence by (2.3.4)

$$S^* = S_N \,\widehat{+}\, \widehat{\mathfrak{N}}_{-a}(S^*).$$

In particular,

$$\operatorname{dom} S^* = \operatorname{dom} S_N + \ker\,(S^*+a),$$

and, since $\ker\,(S^*+a) \subset \operatorname{dom} S_N^{1/2}$, the inclusion (5.2.22) follows. ∎

The linear manifold $\ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}$ in Proposition 5.2.4 is intimately connected with the so-called Kac subclass of the class of Nevanlinna functions, cf. [36], [34]. Observe that the decomposition in (5.2.18) is orthogonal when $\operatorname{dom} H^{1/2}$ is provided with the graph inner product relative to $H^{1/2}$. In the context of nondensely defined operators (or relations) this result goes back to [20]. Corollary 5.2.6 is due to M.M. Malamud [43], who gave a proof involving boundary triplets and Kreĭn's formula; see also Proposition 5.4.4.

5.3 A factorization of the extreme extensions

Let S be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. It is not assumed that S is closed or that its domain of definition $\operatorname{dom} S$ is dense in $\mathfrak{H}$. In this section the fundamental factorizations of the Kreĭn-von Neumann and the Friedrichs extensions of S are established.

Provide the linear space $\operatorname{ran} S$ with a semi-inner product $\langle \cdot, \cdot \rangle$ by

$$\langle f', g' \rangle := (f', g) = (f, g'), \quad \{f, f'\}, \{g, g'\} \in S. \tag{5.3.1}$$

Note that if also $\{f_0, f'\}, \{g_0, g'\} \in S$, then the symmetry of S implies that

$$(f', g) = (f, g') = (f', g_0) = (f_0, g'), \tag{5.3.2}$$

which shows that the inner product (5.3.1) is well defined. Define the linear space $\mathfrak{R}_0$ by

$$\mathfrak{R}_0 = \{\, f' \,:\, (f', f) = 0 \quad \text{for some } \{f, f'\} \in S \,\}. \tag{5.3.3}$$

Note that if $(f', f) = 0$ for $\{f, f'\} \in S$, then also $(f', f_0) = 0$ when $\{f_0, f'\} \in S$, cf. (5.3.2). In general, the space $\mathfrak{R}_0$ is nontrivial. Clearly the definition implies that $\operatorname{mul} S \subset \mathfrak{R}_0 \subset \operatorname{ran} S$.

Lemma 5.3.1. *Let S be a nonnegative relation. Then*

$$\mathfrak{R}_0 = \operatorname{ran} S \cap \operatorname{mul} S^*. \tag{5.3.4}$$

Proof. Assume that $f' \in \operatorname{ran} S \cap \operatorname{mul} S^*$. Then $\{f, f'\} \in S$ for some $f \in \mathfrak{H}$, and $\{0, f'\} \in S^*$. This implies that $(f', f) = 0$ and therefore $f' \in \mathfrak{R}_0$.

Conversely, assume that $f' \in \mathfrak{R}_0$. Then $(f', f) = 0$ for some $\{f, f'\} \in S$. Clearly with $\{g, g'\} \in S$ the Cauchy-Schwarz inequality (4.2.4) gives

$$|(f', g)|^2 \le (f', f)(g', g) = 0.$$

Hence $(f', g) = 0$, so that $f' \in (\operatorname{dom} S)^\perp = \operatorname{mul} S^*$. This implies that $f' \in \operatorname{ran} S \cap \operatorname{mul} S^*$. ■

The quotient space $\operatorname{ran} S / \mathfrak{R}_0$ equipped with the inner product

$$\langle [f'], [g'] \rangle := (f', g) = (f, g'), \quad \{f, f'\}, \{g, g'\} \in S, \tag{5.3.5}$$

where $[f']$, $[g']$ denote the equivalence classes containing f' and g', is a pre-Hilbert space.

Definition 5.3.2. *The Hilbert space completion of $\operatorname{ran} S/\mathfrak{R}_0$ is denoted by $\mathfrak{H}_S$; its inner product is again denoted by $\langle \cdot, \cdot \rangle$. The linear relation Q from $\mathfrak{H}$ to $\mathfrak{H}_S$ is defined by*

$$Q = \{\, \{f, [f']\} : \{f, f'\} \in S \,\}. \tag{5.3.6}$$

The linear relation J from $\mathfrak{H}_S$ to $\mathfrak{H}$ is defined by

$$J = \{\, \{[f'], f'\} : \{f, f'\} \in S \,\}. \tag{5.3.7}$$

Note that $\operatorname{mul} Q = \{0\}$, i.e., Q is (the graph of) an operator, and that $\operatorname{dom} Q = \operatorname{dom} S$. To see that $\operatorname{mul} Q = \{0\}$, assume that $f = 0$ in (5.3.6). Then $\{0, f'\} \in S$ and, by (5.3.3), clearly $f' \in \mathfrak{N}_0$, which shows that $[f'] = 0$. Moreover, note that J is multivalued with $\operatorname{mul} J = \mathfrak{N}_0$ and that $\operatorname{ran} J = \operatorname{ran} S$. The relation J is densely defined in $\mathfrak{H}_S$ and therefore J^* is an operator. The definitions (5.3.6) and (5.3.7) imply that

$$J \subset Q^*, \quad Q \subset J^*, \tag{5.3.8}$$

as follows from (5.3.5). In particular, $Q^{**} = \operatorname{clos} Q$ is a restriction of J^*, so that Q^{**}, the closure of the operator Q, is also an operator. Recall that the product relations $J^{**}J^*$ and Q^*Q^{**} are nonnegative and selfadjoint, cf. Chapter 4. Since both J^* and Q^{**} are operators, the associated nonnegative forms are defined on $\operatorname{dom} J^*$ and $\operatorname{dom} Q^{**}$, respectively, cf. Chapter 4. The next theorem extends [10, Proposition 3.1]; here S need not be densely defined and is even allowed to be a nonnegative (not necessarily closed) relation.

Theorem 5.3.3. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (5.3.6) *and* (5.3.7)*. Then the Kreĭn-von Neumann extension S_N of S is given by $S_N = J^{**}J^*$ and the corresponding closed form $\mathfrak{t}_N$ is given by*

$$\mathfrak{t}_N[f,g] = \langle J^*f, J^*g\rangle\,, \quad f,\, g \in \operatorname{dom} J^* = \operatorname{dom} S_N^{1/2}.$$

Furthermore, the Friedrichs extension S_F of S is given by $S_F = Q^*Q^{**}$ and the corresponding closed form $\mathfrak{t}_F$ is given by

$$\mathfrak{t}_F[f,g] = \langle Q^{**}f, Q^{**}g\rangle\,, \quad f,\, g \in \operatorname{dom} Q^{**} = \operatorname{dom} S_F^{1/2}.$$

Proof. First consider the case of the Kreĭn-von Neumann extension. Let $\{f, f'\} \in S$. Since $\{f, [f']\} \in Q \subset J^*$ and $\{[f'], f'\} \in J \subset J^{**}$ it follows that $\{f, f'\} \in J^{**}J^*$, i.e. $S \subset J^{**}J^*$. Hence the nonnegative selfadjoint relation $J^{**}J^*$ is an extension of S.

Now let $\{f, f'\} \in J^{**}J^*$. Then $\{f, J^*f\} \in J^*$ and $\{J^*f, f'\} \in J^{**}$. The identity $J^{**} = \operatorname{clos} J$ implies the existence of a sequence $(\{[f_n'], f_n'\}) \subset J$, where $\{f_n, f_n'\} \in S$, such that

$$[f_n'] \to J^*f \text{ in } \mathfrak{H}_S, \quad f_n' \to f' \text{ in } \mathfrak{H}. \tag{5.3.9}$$

It follows from $\{[f_n'], f_n'\} \in J$ and $\{f, J^*f\} \in J^*$ that

$$\langle [f_n'], J^*f\rangle = (f_n', f). \tag{5.3.10}$$

Likewise, it follows from $\{f, J^*f\} \in J^*$ and $\{J^*f, f'\} \in J^{**}$ that

$$\langle J^*f, J^*f\rangle = (f, f'). \tag{5.3.11}$$

Finally note that $\{f_n, f_n'\} \in S \subset J^{**}J^*$ and $\{f, f'\} \in J^{**}J^*$ imply that

$$(f', f_n) = (f, f_n'). \tag{5.3.12}$$

This leads to the following identity

$$\begin{aligned}
\langle [f_n'] - J^* f \quad &, \quad [f_n'] - J^* f \rangle \\
&= \langle [f_n'], [f_n'] \rangle - \langle [f_n'], J^* f \rangle - \langle J^* f, [f_n'] \rangle + \langle J^* f, J^* f \rangle \\
&= (f_n', f_n) - (f_n', f) - (f, f_n') + (f, f') \\
&= (f_n', f_n) - (f_n', f) - (f', f_n) + (f', f) \\
&= (f' - f_n', f - f_n),
\end{aligned}$$

where (5.3.10), (5.3.11), and (5.3.12) have been used, respectively. Therefore (5.3.9) implies that

$$f_n' \to f' \text{ in } \mathfrak{H}, \quad (f' - f_n', f - f_n) \to 0. \tag{5.3.13}$$

Since $\{f_n, f_n'\} \in S$, this shows that $\{f, g\} \in S_N$, cf. Proposition 5.2.1. Hence, $J^{**}J^* \subset S_N$, and since $J^{**}J^*$ and S_N are both selfadjoint, the identity $J^{**}J^* = S_N$ follows. The statement concerning the associated form $\mathfrak{t}_N$ follows from Proposition 4.5.2.

Next consider the case of the Friedrichs extension. Let $\{f, f'\} \in S$. Since $\{f, [f']\} \in Q \subset Q^{**}$ and $\{[f'], f'\} \in J \subset Q^*$ it follows that $\{f, f'\} \in Q^*Q^{**}$, i.e. $S \subset Q^*Q^{**}$. Thus the nonnegative selfadjoint relation Q^*Q^{**} is an extension of S.

Now let $\{f, f'\} \in Q^*Q^{**}$. Then $\{f, Q^{**}f\} \in Q^{**}$ and $\{Q^{**}f, f'\} \in Q^*$. The identity $Q^{**} = \operatorname{clos} Q$ implies the existence of a sequence $(\{f_n, [f_n']\}) \subset Q$ where $\{f_n, f_n'\} \in S$, such that

$$f_n \to f \text{ in } \mathfrak{H}, \quad [f_n'] \to Q^{**}f \text{ in } \mathfrak{H}_S. \tag{5.3.14}$$

It follows from $\{[f_n'], f_n'\} \in J \subset Q^*$, and $\{f, Q^{**}f\} \in Q^{**}$ that

$$\langle [f_n'], Q^{**}f \rangle = (f_n', f). \tag{5.3.15}$$

Likewise, it follows from $\{f, Q^{**}f\} \in Q^{**}$ and $\{Q^{**}f, f'\} \in Q^*$ that

$$\langle Q^{**}f, Q^{**}f \rangle = (f, f'). \tag{5.3.16}$$

Finally, note that $\{f_n, f_n'\} \in S \subset Q^*Q^{**}$, and $\{f, f'\} \in Q^*Q^{**}$ imply that

$$(f', f_n) = (f, f_n'). \tag{5.3.17}$$

This leads to the following identity

$$\begin{aligned}
\langle [f_n'] - Q^{**} f \quad &, \quad [f_n'] - Q^{**} f \rangle \\
&= \langle [f_n'], [f_n'] \rangle - \langle [f_n'], Q^{**} f \rangle - \langle Q^{**} f, [f_n'] \rangle + \langle Q^{**} f, Q^{**} f \rangle \\
&= (f_n', f_n) - (f_n', f) - (f, f_n') + (f, f') \\
&= (f_n', f_n) - (f_n', f) - (f', f_n) + (f', f) \\
&= (f' - f_n', f - f_n),
\end{aligned}$$

where (5.3.15), (5.3.16), and (5.3.17) have been used, respectively. Therefore (5.3.14) implies that

$$f_n \to f \text{ in } \mathfrak{H}, \quad (f' - f'_n, f - f_n) \to 0. \tag{5.3.18}$$

Since $\{f_n, f'_n\} \in S$, this shows that $\{f, f'\} \in S_F$, cf. Proposition 5.2.1. Hence, $Q^*Q^{**} \subset S_F$, and since Q^*Q^{**} and S_F are both selfadjoint, the identity $Q^*Q^{**} = S_F$ follows. Again the statement concerning the associated form $\mathfrak{t}_F$ follows from Proposition 4.5.2. ∎

The factorizations of the Kreĭn-von Neumann and the Friedrichs extensions in Theorem 5.3.3, in the case where these extensions are operators, go back to Z. Sebestyén, J. Stochel, and coworkers, see [50], [49], [57], [59], [58]. The case of densely defined nonnegative operators was treated in [10], where also the connection with the corresponding closed nonnegative forms was given. The context of non-densely defined operators and, more generally, of relations requires the introduction of equivalence classes and the space $\mathfrak{R}_0$ in order to define the relation J. However, it is implicit in the construction that the adjoint J^* is (the graph of) an operator. This fact makes it possible to describe the forms associated with the Kreĭn-von Neumann and the Friedrichs extensions in terms of J^* and its restriction $Q^{**} \subset J^*$.

5.4 Some consequences of the factorizations

The factorizations of the Kreĭn-von Neumann and the Friedrichs extensions in Theorem 5.3.3 appear to be natural tools to study various questions concerning nonnegative selfadjoint extensions of nonnegative operators or relations. In particular, in this section the disjointness, the transversality, and the equality of the Kreĭn-von Neumann and the Friedrichs extensions are characterized by means of these factorizations. Also criteria for the Kreĭn-von Neumann extension to be an operator will be developed in terms of its factorization.

5.4.1 Some general observations about factorizations

The intersection of the Kreĭn-von Neumann extension S_N and the Friedrichs extension S_F is a closed nonnegative relation which extends S:

$$S \subset S_N \cap S_F. \tag{5.4.1}$$

Clearly, it follows from (5.4.1) that

$$S_N \mathbin{\widehat{+}} S_F \subset \operatorname{clos}(S_N \mathbin{\widehat{+}} S_F) \subset S^*. \tag{5.4.2}$$

It will be of interest to characterize the situation when there is equality instead of inclusion in (5.4.1) and (5.4.2). Furthermore, the other extremal case, where the Kreĭn-von Neumann extension and the Friedrichs extension coincide, will be characterized.

Lemma 5.4.1. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (5.3.6) *and* (5.3.7)*. Then*

$$S_N \cap S_F = J^{**}Q^{**}, \tag{5.4.3}$$

and, in particular,

$$\operatorname{mul}(S_N \cap S_F) = \operatorname{mul} J^{**}. \tag{5.4.4}$$

Proof. It follows from (5.3.8) that

$$Q \subset Q^{**} \subset J^*, \quad J \subset J^{**} \subset Q^*,$$

and therefore Theorem 5.3.3 shows that

$$J^{**}Q^{**} \subset Q^*Q^{**} \cap J^{**}J^* = S_N \cap S_F.$$

Thus, $J^{**}Q^{**} \subset S_N \cap S_f$. To prove the reverse inclusion, let $\{f, f'\} \in S_F \cap S_N$. This implies that $\{f, h\} \in Q^{**}$, $\{h, f'\} \in Q^*$ for some $h \in \mathfrak{H}_S$, and that $\{f, k\} \in J^*$, $\{k, f'\} \in J^{**}$ for some $k \in \mathfrak{H}_S$. Observe that $\{f, h\} \in Q^{**} \subset J^*$ and since J^* is an operator, one concludes that $k = h$. Hence, $\{f, f'\} \in J^{**}Q^{**}$.

Since $\operatorname{mul} Q^{**} = \{0\}$, $\operatorname{mul} J^{**}Q^{**} = \operatorname{mul} J^{**}$ and hence the equality (5.4.4) follows immediately from (5.4.3). ■

Closely related to the closed linear relation $J^{**}Q^{**}$ in $\mathfrak{H}$ is the relation Q^*J^*, also in $\mathfrak{H}$, which is not necessarily closed.

Lemma 5.4.2. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (5.3.6) *and* (5.3.7)*. Then*

$$S_N \mathbin{\widehat{+}} S_F \subset Q^*J^* \subset S^*, \tag{5.4.5}$$

and, in particular,

$$S \subset (Q^*J^*)^* \subset S_N \cap S_F. \tag{5.4.6}$$

Proof. Recall that $Q^{**} \subset J^*$ and $J^{**} \subset Q^*$. Hence

$$S_F = Q^*Q^{**} \subset Q^*J^*, \quad S_N = J^{**}J^* \subset Q^*J^*,$$

so that $S_N \mathbin{\widehat{+}} S_F \subset Q^*J^*$. Next it is shown that $Q^*J^* \subset S^*$. Assume that $\{f, f'\} \in Q^*J^*$, so that $\{f, \varphi\} \in J^*$ and $\{\varphi, f'\} \in Q^*$ for some $\varphi \in \mathfrak{H}_S$. By definition, for any $\{h, h'\} \in S$ one has

$$\{[h'], h'\} \in J, \quad \{h, [h']\} \in Q,$$

cf. (5.3.6) and (5.3.7). This leads to

$$(f, h') = \langle \varphi, [h'] \rangle, \quad (f', h) = \langle \varphi, [h'] \rangle,$$

which implies $(f', h) = (f, h')$, and hence $\{f, f'\} \in S^*$. This completes the proof of (5.4.5). The inclusions in (5.4.6) follow by taking adjoints in (5.4.5). ■

5.4.2 Disjointness and transversality of the Kreĭn-von Neumann and the Friedrichs extensions

The operator Q^*J^* plays an important role in the description of the disjointness and transversality of S_N and S_F.

Proposition 5.4.3. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (5.3.6) *and* (5.3.7). *Then the following statements are equivalent:*

(i) *S_N and S_F are disjoint;*

(ii) *$S = J^{**}Q^{**}$;*

(iii) *$S = (Q^*J^*)^*$;*

(iv) *S is closed and $S^* = \operatorname{clos}(Q^*J^*)$.*

Proof. The equivalence between (i) and (ii) is clear from Lemma 5.4.1. Furthermore, the equivalence between (iii) and (iv) is obvious.

(i) $\Rightarrow$ (iii) Since S_N and S_F are disjoint, so that $S_N \cap S_F = S$, the statement follows from (5.4.6).

(iii) $\Rightarrow$ (ii) Assume that $S = (Q^*J^*)^*$, then by (2.4.17) and by Lemma 5.4.1 it follows that

$$S = (Q^*J^*)^* \supset J^{**}Q^{**} = S_N \cap S_F \supset S,$$

so that S is disjoint. ■

Proposition 5.4.4. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (5.3.6) *and* (5.3.7). *Then the following statements are equivalent:*

(i) *S_N and S_F are transversal;*

(ii) *$S^* = Q^*J^*$;*

(iii) $\operatorname{dom} S^* \subset \operatorname{dom} J^*$*;*

(iv) $\ker\,(S^* + a) \subset \operatorname{dom} J^*$ *for some (and hence for all) $a > 0$.*

Proof. (i) $\Rightarrow$ (ii) Assume that S_N and S_F are transversal. Then it follows from (5.4.5) that $S^* = Q^*J^*$, i.e., (ii) is valid.

(ii) $\Rightarrow$ (iii) & (iii) $\Rightarrow$ (iv) These implications are trivial.

(iv) $\Rightarrow$ (i) Assume that $\ker\,(S^* + a) \subset \operatorname{dom} J^*$. By Theorem 5.3.3 $\operatorname{dom} J^* = \operatorname{dom} S_N^{1/2}$ and the transversality of S_N and S_F follows from Proposition 5.2.5. ■

5.4.3 Equality of the Kreĭn-von Neumann and the Friedrichs extensions

The factorizations can also be used to determine when the Kreĭn-von Neumann and the Friedrichs extensions coincide. In the present context one simple criterion is immediate.

Proposition 5.4.5. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (5.3.6) *and* (5.3.7)*. Then $S_N = S_F$ if and only if $J^* = Q^{**}$.*

Proof. If $J^* = Q^{**}$, then $J^{**} = Q^*$, so that $J^{**}J^* = Q^*Q^{**}$, which is equivalent to $S_F = S_N$ by Theorem 5.3.3.

Conversely, assume that $S_N = S_F$, or equivalently, $J^{**}J^* = Q^*Q^{**}$. Then $\operatorname{dom} J^* = \operatorname{dom} Q^{**}$ and since $Q^{**} \subset J^*$ and J^* is an operator, one obtains that $J^* = Q^{**}$. ∎

In order to obtain an analytical uniqueness criterion the following observation is useful.

Proposition 5.4.6. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then*

$$\operatorname{dom} J^* = \left\{ g \in \mathfrak{H} : |(f', g)|^2 \le C_g(f', f) \text{ for all } \{f, f'\} \in S \text{ and some } C_g < \infty \right\}.$$

Proof. Recall that J is the relation from $\mathfrak{H}_S$ to $\mathfrak{H}$ defined in (5.3.7). Apply the description (2.4.1) in Lemma 2.4.3 to J to obtain a description of $\operatorname{dom} J^*$: $g \in \operatorname{dom} J^*$ if and only if $g \in \mathfrak{H}$ and there exists a nonnegative number γ_g such that

$$|(f', g)| \le \gamma_g \|[f']\|_{\mathfrak{H}_S} \quad \text{for all} \quad \{[f'], f'\} \in J. \tag{5.4.7}$$

By (5.3.7) and (5.3.5) the estimate in (5.4.7) can be rewritten as

$$|(f', g)|^2 \le \gamma_g^2 (f', f) \quad \text{for all} \quad \{f, f'\} \in S. \tag{5.4.8}$$

This gives the description of $\operatorname{dom} J^*$. ∎

The description of $\operatorname{dom} J^*$ leads to the uniqueness criterion due to Kreĭn [41].

Theorem 5.4.7 (Kreĭn's uniqueness criterion). *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then $S_N = S_F$ if and only if for some (and hence for all) $a > 0$*

$$\sup \left\{ \frac{|(f, \varphi)|^2}{(f', f)} : \{f, f'\} \in S \right\} = \infty \quad \textit{for all} \quad \varphi \in \ker\,(S^* + a) \setminus \{0\}. \tag{5.4.9}$$

Proof. By Proposition 5.2.5 with $H = S_N$, one has $S_N = S_F$ if and only if

$$\ker\,(S^* + a) \cap \operatorname{dom} S_N^{1/2} = \{0\}. \tag{5.4.10}$$

In other words $S_N = S_F$ if and only if

$$\varphi \in \ker\,(S^*+a)\setminus\{0\} \Rightarrow \varphi \notin \operatorname{dom} S_N^{1/2}. \tag{5.4.11}$$

By Proposition 5.4.6 and Theorem 5.3.3 it follows that $\varphi \in \operatorname{dom} S_N^{1/2} = \operatorname{dom} J^*$ if and only if there is a nonnegative number C_φ such that

$$|(f',\varphi)|^2 \le C_\varphi (f',f) \quad \text{for all} \quad \{f,f'\}\in S.$$

Since $\varphi \in \ker\,(S^*+a)$ it is clear that $(f',\varphi) = a(f,\varphi)$. Hence, $\varphi \in \operatorname{dom} S_N^{1/2}$ if and only if

$$|(f,\varphi)|^2 \le \frac{C_\varphi}{a}(f',f) \quad \text{for all} \quad \{f,f'\}\in S,$$

and this leads to the description (5.4.9). ■

The description of the form domain $\operatorname{dom} \mathfrak{t}_N = \operatorname{dom} J^*$ in Theorem 5.3.3 is made explicit in Proposition 5.4.6. This description coincides with a result of Ando and Nishio [2] for the class of positively closable operators. Kreĭn's uniqueness criterion was originally stated for the case of densely defined operators; its formulation for nonnegative relations can be found in [37].

5.4.4 Positive closability

The Friedrichs extension S_F is an operator if and only if S is densely defined. In this case, each nonnegative selfadjoint extension H of S is an operator. However, when S is not densely defined, there may be nonnegative selfadjoint extensions of S which are operators, in which case S is automatically an operator. In what follows, a relation in a Hilbert space $\mathfrak{H}$ is called *closable* if its closure in the Cartesian product $\mathfrak{H}\times\mathfrak{H}$ is (the graph of) an operator. Of course, a closable relation is itself already an operator.

Proposition 5.4.8. *Let S be a nonnegative relation. Then*

$$\operatorname{mul} S_N = \operatorname{mul} J^{**} \subset \overline{\operatorname{ran}}\, S \cap \operatorname{mul} S^*. \tag{5.4.12}$$

Furthermore, the following statements are equivalent:

(i) *S_N is an operator;*

(ii) *S has a nonnegative selfadjoint operator extension;*

(iii) *the relation J is closable;*

(iv) *$S_N \cap S_F$ is an operator.*

In this case S is a nonnegative operator.

Proof. By Theorem 5.3.3 $S_N = J^{**}J^*$, which gives the identity in (5.4.12). For the inclusion in (5.4.12), recall that $J \subset Q^*$ and $\mathrm{dom}\, Q = \mathrm{dom}\, S$, so that

$$\mathrm{mul}\, J^{**} \subset \mathrm{mul}\, Q^* = \mathrm{mul}\, S^*.$$

Furthermore, observe that $\mathrm{mul}\, J^{**} \subset \mathrm{ran}\, J^{**} \subset \overline{\mathrm{ran}}\, J$ and $\mathrm{ran}\, J = \mathrm{ran}\, S$, so that

$$\mathrm{mul}\, J^{**} \subset \overline{\mathrm{ran}}\, S.$$

Hence also the inclusion in (5.4.12) has been proved.

(i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (i) Let H be a nonnegative selfadjoint operator extension of S (so that also S is an operator), then it follows from (5.1.7) that S_N is an operator.

(i) $\Leftrightarrow$ (iii) According to (5.4.12) S_N is an operator if and only if $\mathrm{mul}\, J^{**}$ is trivial.

(i) $\Leftrightarrow$ (iv) This follows immediately from (5.4.4), which leads to $\mathrm{mul}\,(S_N \cap S_F) = \mathrm{mul}\, S_N$. ■

Corollary 5.4.9. *Let S be a nonnegative relation in $\mathfrak{H}$. Then*

$$\mathrm{mul}\, S_N = \{\, f' \in \mathfrak{H} : \{f_n, f_n'\} \in S,\ f_n' \to f',\ (f_n', f_n) \to 0 \,\}. \tag{5.4.13}$$

In particular, S_N is an operator if and only if

$$\{f_n, f_n'\} \in S,\ \lim_{n\to\infty} (f_n', f_n) = 0,\ \textit{and}\ \lim_{n\to\infty} f_n' = f'\ \textit{imply}\ f' = 0. \tag{5.4.14}$$

Proof. By Proposition 5.4.8 one has $\mathrm{mul}\, S_N = \mathrm{mul}\, J^{**}$ and hence the definitions of $\mathfrak{H}_S$ and J, see (5.3.1), (5.3.3), (5.3.7), show that the multivalued part of the closure J^{**} of J is given by (5.4.13). Therefore, S_N is an operator if and only if (5.4.14) is satisfied. This completes the proof. ■

A nonnegative relation S is said to be *positively closable* if it satisfies the property (5.4.14). In this case, S is automatically closable (as $S \subset S_N = S_N^*$) and thus an operator. Hence the Kreĭn-von Neumann extension S_N is an operator if and only if S is positively closable.

Corollary 5.4.10. *Let S be a nonnegative relation in $\mathfrak{H}$. Then*

(i) *if* $\mathrm{ran}\, S$ *is closed, then* $S_N = S \,\widehat{+}\, (\ker\, S^* \times \{0\})$ *and* $\mathrm{mul}\, S_N = \mathrm{mul}\, J$;

(ii) *if* $m(S) > 0$, *then* $\mathrm{mul}\, S_N = \mathrm{mul}\, \mathrm{clos}\, S$.

Proof. (i) Since $\mathrm{ran}\, S$ is closed it follows from (5.2.7) that S_N has the form (5.2.6). By Proposition 5.4.8 and (5.3.4) one has

$$\mathrm{mul}\, S_N \subset \overline{\mathrm{ran}}\, S \cap \mathrm{mul}\, S^* = \mathrm{ran}\, S \cap \mathrm{mul}\, S^* = \mathrm{mul}\, J,$$

and, since $\mathrm{mul}\, J \subset \mathrm{mul}\, S_N$, this gives $\mathrm{mul}\, S_N = \mathrm{mul}\, J$.

(ii) The inclusion $\mathrm{mul}\, \mathrm{clos}\, S \subset \mathrm{mul}\, S_N$ is clear. The converse follows from (5.4.13) since $(f_n', f_n) \geq m(S)\|f_n\|^2$ for $\{f_n, f_n'\} \in S$. ■

In other words, if $\operatorname{ran} S$ is closed, then S is positively closable if and only if $\operatorname{mul} J = \{0\}$, and if $m(S) > 0$, then S is positively closable if and only if S is closable.

Proposition 5.4.11. *Let S be a nonnegative operator. Then:*

(i) *if S_N and S_F are disjoint, then S_N is an operator;*

(ii) *if S_N is an operator and if S satisfies the property* (5.2.9)*, then S_N and S_F are disjoint.*

In particular, if S is a nonnegative closed bounded operator, then S_N is an operator if and only if S_N and S_F are disjoint.

Proof. (i) Assume that $S_N \cap S_F = S$. Let $\{0, \varphi\} \in S_N$. Then clearly $\{0, \varphi\} \in S_F$, so that $\{0, \varphi\} \in S_N \cap S_F \subset S$. Therefore $\varphi = 0$, and S_N is an operator.

(ii) Let S be a nonnegative operator which satisfies the property (5.2.9) and let S_N be an operator. Assume that $\{f, f'\} \in S_N \cap S_F$. Since $\{f, f'\} \in S_F$, it follows from (5.2.8) that $\{f, f'\} \in S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*)$. Hence $\{f, f'\} = \{h, Sh\} \,\widehat{+}\, \{0, \varphi\}$ with $h \in \operatorname{dom} S$ and $\varphi \in \operatorname{mul} S^*$. Therefore $\{0, \varphi\} \in S_N \cap S_F$ and since S_N is an operator, one concludes that $\varphi = 0$. Hence $\{f, f'\} = \{h, Sh\} \in S$, which shows $S_N \cap S_F \subset S$.

If S is a bounded closed nonnegative operator, then $\operatorname{dom} S$ is closed, so that the condition (5.2.9) is satisfied. ■

Observe that the conditions in (ii) of Proposition 5.4.11 automatically imply that S is closed. Furthermore, if S_N is an operator and the condition (5.2.9) is not satisfied, then S_N and S_F need not be disjoint: there are densely defined nonnegative nonselfadjoint operators S for which S_N and S_F coincide, cf. Theorem 5.4.7.

Now the problem arises to determine, parallel to Proposition 5.4.8, the nonnegative linear relations S whose Kreĭn-von Neumann extension S_N is a bounded operator.

Proposition 5.4.12. *Let S be a nonnegative relation. Then the following statements are equivalent:*

(i) *S_N belongs to $[\mathfrak{H}]$;*

(ii) *S has at least one nonnegative extension in $[\mathfrak{H}]$;*

(iii) *$\|f'\|^2 \leq M(f', f)$ for all $\{f, f'\} \in S$ and for some $M \geq 0$;*

(iv) *J is a bounded operator.*

In this case S is a nonnegative bounded operator.

Proof. (i) $\Rightarrow$ (ii) This implication is clear.

(ii) $\Rightarrow$ (iii) Let H be a nonnegative operator in $[\mathfrak{H}]$ which extends S. Then clearly $\|H^{1/2}(H^{1/2}f)\| \leq \|H^{1/2}\| \, \|H^{1/2}f\|$, so that

$$\|Hf\|^2 \leq \|H\| \, (Hf, f), \quad f \in \operatorname{dom} H.$$

Restricting f to $\operatorname{dom} S$ one gets the inequality in (iii) with $M = \|H\|$.

(iii) $\Rightarrow$ (iv) In view of (5.3.3) one has $\mathfrak{N}_0 = \{0\}$ and hence the relation J is (the graph of) an operator. The boundedness of J is also a consequence of the inequality.

(iv) $\Rightarrow$ (i) Let J be a bounded operator. Then its closure J^{**} is a bounded operator with the same norm. Furthermore, since J is densely defined, the bounded operator J^{**} is defined everywhere. Hence $S_N = J^{**}J^* \in [\mathfrak{H}]$. ∎

Furthermore the following proposition is parallel to Proposition 5.4.11.

Proposition 5.4.13. *Let S be a bounded nonnegative operator. Then $S_N \in [\mathfrak{H}]$ if and only if S_N and S_F are transversal.*

Proof. Since the operator S is bounded, it is closable, i.e., $\operatorname{mul} \operatorname{clos} S = \{0\}$. Hence $\operatorname{dom} S^* = (\operatorname{mul} \operatorname{clos} S)^\perp = \mathfrak{H}$.

Assume that $S_N \in [\mathfrak{H}]$. Then J is a bounded operator, densely defined, so that J^* is a bounded operator, cf. Proposition 5.4.12. Therefore $\operatorname{dom} S^* = \mathfrak{H} = \operatorname{dom} J^*$. Hence S_N and S_F are transversal by Proposition 5.4.4.

Conversely, assume that S_N and S_F are transversal. Hence $\mathfrak{H} = \operatorname{dom} S^* \subset \operatorname{dom} J^*$, which implies that J^* is a bounded operator. Therefore J is a bounded operator, and hence $S_N \in [\mathfrak{H}]$. ∎

The notion of positive closability, introduced by Ando and Nishio [2] for closed nonnegative operators, is inherent in the representation of the Kreĭn-von Neumann extension S_N in Theorem 5.3.3. The connection between the identities (5.4.13) and (5.4.14) was first noticed in [37]. The inequality in part (iii) of Proposition 5.4.12 guarantees that S is an operator. Hence it can be rewritten as $\|Sf\|^2 \le M(Sf, f)$, $f \in \operatorname{dom} S$, and in this form it goes back to Ando and Nishio [2]. As the proof of Proposition 5.4.12 shows, this inequality implies that the mapping J from $\mathfrak{H}_S$ to $\mathfrak{H}$ is bounded with the norm $\|J\| \le \sqrt{M}$, cf. [38]. For Proposition 5.4.13 see also [24].

5.5 A factorization of the extremal extensions of nonnegative relations

In this section the characterization of the extremal extensions of an arbitrary nonnegative relation in terms of factorizations is established. This extends the treatment for densely defined nonnegative operators in [10]. In Section 5.6 the present results will be made explicit when the nonnegative symmetric relation is actually a bounded nonnegative operator.

Let S be a nonnegative relation in $\mathfrak{H}$ and let $\mathfrak{L}$ be any subspace such that

$$\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} J^* = \operatorname{dom} S_N^{1/2}. \tag{5.5.1}$$

Associate with $\mathfrak{L}$ the restriction operator $R_\mathfrak{L}$ from $\mathfrak{H}$ to $\mathfrak{H}_S$ by

$$R_\mathfrak{L} := J^* \upharpoonright \mathfrak{L} = \{\, \{f, f'\} \in J^* \,:\, f \in \mathfrak{L} \,\}\,. \tag{5.5.2}$$

Since J^* is a closed operator from $\mathfrak{H}$ to $\mathfrak{H}_S$, it is clear that $R_{\mathfrak{L}}$ is a closable operator. The definition of $R_{\mathfrak{L}}$ and Theorem 5.3.3 lead to

$$\langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g \rangle = \langle J^* f, J^* g \rangle = \mathfrak{t}_N[f, g], \quad f,\, g \in \mathfrak{L}. \tag{5.5.3}$$

Hence, $R_{\mathfrak{L}}$ is closed if and only if the restriction of the form $\mathfrak{t}_N[\cdot,\cdot]$ to $\mathfrak{L}$ is closed, cf. [40]. Clearly, operators of the form $R_{\mathfrak{L}}$ induce nonnegative selfadjoint relations $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ and corresponding closed nonnegative forms $\mathfrak{t}_{\mathfrak{L}}$ defined by

$$\mathfrak{t}_{\mathfrak{L}}[f, g] = \langle R_{\mathfrak{L}}^{**} f, R_{\mathfrak{L}}^{**} g \rangle \quad f, g \in \operatorname{dom} R_{\mathfrak{L}}^{**}. \tag{5.5.4}$$

Assume that the linear, not necessarily closed, subspaces $\mathfrak{L}$ and $\mathfrak{M}$ satisfy

$$\operatorname{dom} S \subset \mathfrak{L} \subset \mathfrak{M} \subset \operatorname{dom} S_N^{1/2}.$$

Then the closed forms induced by $\mathfrak{L}$ and $\mathfrak{M}$ satisfy the inclusion $\mathfrak{t}_{\mathfrak{L}} \subset \mathfrak{t}_{\mathfrak{M}}$. In particular, $\mathfrak{t}_{\mathfrak{L}} \geq \mathfrak{t}_{\mathfrak{M}}$ which leads to the following monotonicity property (cf. [10] and Chapter 4):

$$R_{\mathfrak{M}}^* R_{\mathfrak{M}}^{**} \leq R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}. \tag{5.5.5}$$

The nonnegative selfadjoint extensions of S which are extremal can be characterized in terms of nonnegative selfadjoint factorizations $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ induced by the operators $R_{\mathfrak{L}}$ in (5.5.2).

Theorem 5.5.1. *Let S be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ *for some $\mathfrak{L}$ such that* $\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_N^{1/2}$*;*

(ii) *$\widetilde{A}$ is a nonnegative selfadjoint extremal extension of S;*

(iii) *$\widetilde{A}$ is a nonnegative selfadjoint extension of S whose associated closed form $\widetilde{\mathfrak{t}}$ satisfies $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_N$.*

Proof. (i) $\Rightarrow$ (ii) Let $\{f, f'\} \in S$. Since $\{f, [f']\} \in Q \subset J^* \upharpoonright \mathfrak{L} = R_{\mathfrak{L}} \subset R_{\mathfrak{L}}^{**}$ and $\{[f'], f'\} \in J \subset J^{**} \subset R_{\mathfrak{L}}^*$, it follows that $\{f, f'\} \in R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$, i.e., $S \subset R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$. Hence, the nonnegative selfadjoint relation $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ is an extension of S.

Let $\{f, f'\} \in \widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$. Then $\{f, J^* f\} \in R_{\mathfrak{L}}^{**}$, and $\{J^* f, f'\} \in R_{\mathfrak{L}}^*$. Therefore,

$$(f', f) = \langle J^* f, J^* f \rangle. \tag{5.5.6}$$

Let $\{h, h'\} \in S$. It follows from $\{J^* f, f'\} \in R_{\mathfrak{L}}^*$ and $\{h, [h']\} \in Q \subset R_{\mathfrak{L}}$, that

$$(f', h) = \langle J^* f, [h'] \rangle. \tag{5.5.7}$$

Furthermore, it follows from $\{f, J^* f\} \in J^*$ and $\{[h'], h'\} \in J$, that

$$(h', f) = \langle [h'], J^* f \rangle. \tag{5.5.8}$$

Finally, note that by definition,

$$(h', h) = \langle [h'], [h'] \rangle. \tag{5.5.9}$$

The identities (5.5.6)–(5.5.9) lead to

$$\begin{aligned} (f' - h', f - h) &= (f', f) - (f', h) - (h', f) + (h', h) \qquad (5.5.10)\\ &= \langle J^* f, J^* f \rangle - \langle J^* f, [h'] \rangle - \langle [h'], J^* f \rangle + \langle [h'], [h'] \rangle \\ &= \| J^* f - [h'] \|^2_{\mathfrak{H}_S}. \end{aligned}$$

The assumption $\{f, f'\} \in \widetilde{A} = R^*_{\mathfrak{L}} R^{**}_{\mathfrak{L}}$ implies that $f \in \operatorname{dom} J^*$, so that $J^* f \in \mathfrak{H}_S$. By definition, $\mathfrak{H}_S$ is the completion of $\operatorname{ran} S/\mathfrak{R}_0$ with respect to the norm $\| \cdot \|_{\mathfrak{H}_S}$. Therefore,

$$\inf \{ \| J^* f - [h'] \|_{\mathfrak{H}_S} \, : \, \{h, h'\} \in S \} = 0,$$

so that (5.5.10) leads to

$$\inf \{ \, (f' - h', f - h) : \, \{h, h'\} \in S \, \} = 0.$$

This shows that the nonnegative selfadjoint extension $\widetilde{A} = R^*_{\mathfrak{L}} R^{**}_{\mathfrak{L}}$ is extremal.

(ii) $\Rightarrow$ (iii) Assume that $\widetilde{A}$ is a nonnegative selfadjoint extension of S and let $\widetilde{\mathfrak{t}}$ be the associated nonnegative closed form. Observe that the inequality $S_N \leq \widetilde{A}$ is equivalent to $\mathfrak{t}_N \leq \widetilde{\mathfrak{t}}$, in other words

$$\operatorname{dom} \widetilde{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}_N, \quad \mathfrak{t}_N[f, f] \leq \widetilde{\mathfrak{t}}[f, f], \quad f \in \operatorname{dom} \widetilde{\mathfrak{t}}, \tag{5.5.11}$$

cf. Theorem 4.4.3. Let $\{f, f'\} \in \widetilde{A}$ and $\{h, h'\} \in S$. It follows from

$$\operatorname{dom} \widetilde{A} \subset \operatorname{dom} \widetilde{A}^{1/2} = \operatorname{dom} \widetilde{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}_N = \operatorname{dom} J^*,$$

that $\{f, J^* f\} \in J^*$. Together with $\{[h'], h'\} \in J$ this leads to

$$(h', f) = \langle [h'], J^* f \rangle. \tag{5.5.12}$$

Furthermore, by definition (see (5.3.1))

$$(h', h) = \langle [h'], [h'] \rangle. \tag{5.5.13}$$

Finally, note that $\{h, h'\}$, $\{f, f'\} \in \widetilde{A}$ imply that

$$(h', f) = (h, f'). \tag{5.5.14}$$

The identities (5.5.12)–(5.5.14) show that for all $\{f, f'\} \in \widetilde{A}$ and $\{h, h'\} \in S$

$$(f' - h', f - h) - \| J^* f - [h'] \|^2_{\mathfrak{H}_S} = (f', f) - \langle J^* f, J^* f \rangle. \tag{5.5.15}$$

Now assume in addition that the nonnegative selfadjoint extension $\widetilde{A}$ of S is extremal. Let $\{f, f'\} \in \widetilde{A}$ and let $\varepsilon > 0$. By (5.1.10) there exists an element $\{h, h'\} \in S$ such that

$$(f' - h', f - h) < \varepsilon. \tag{5.5.16}$$

Clearly $\{f-h, f'-h'\} \in \widetilde{A}$ and by (5.5.11) it follows that

$$\mathfrak{t}_N[f-h, f-h] \leq \widetilde{\mathfrak{t}}[f-h, f-h] = (f'-h', f-h). \tag{5.5.17}$$

Recall that $\{h, h'\} \in S$ implies that $\{h, [h']\} \in Q \subset J^*$, so that according to Theorem 5.3.3

$$\mathfrak{t}_N[f-h, f-h] = \langle J^*(f-h), J^*(f-h)\rangle = \|J^*f - [h']\|^2_{\mathfrak{H}_S}. \tag{5.5.18}$$

A combination of (5.5.16), (5.5.17), and (5.5.18) leads to

$$0 \leq (f'-h', f-h) - \|J^*f - [h']\|^2_{\mathfrak{H}_S} < \varepsilon. \tag{5.5.19}$$

Now combine the inequality (5.5.19) with the identity (5.5.15) to obtain

$$0 \leq (f', f) - \langle J^*f, J^*f\rangle < \varepsilon,$$

where $\{f, f'\} \in \widetilde{A}$ and $\varepsilon > 0$ is arbitrary. Hence

$$(f', f) = \langle J^*f, J^*f\rangle = \mathfrak{t}_N[f, f],$$

for all $\{f, f'\} \in \widetilde{A}$, and, by polarization,

$$(f', g) = \mathfrak{t}_N[f, g], \quad \{f, f'\}, \{g, g'\} \in \widetilde{A}.$$

Therefore the restriction to $\operatorname{dom} \widetilde{A}$ of the form $\widetilde{\mathfrak{t}}$ is a restriction of the form $\mathfrak{t}_N$. Since the form $\mathfrak{t}_N$ is closed, the inclusion $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_N$ holds too.

(iii) $\Rightarrow$ (i) Assume that $\widetilde{A}$ is a nonnegative selfadjoint extension of S such that the closed form $\widetilde{\mathfrak{t}}$ associated to $\widetilde{A}$ satisfies the inclusion $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_N$. Define the subspace $\mathfrak{L}$ by $\mathfrak{L} = \operatorname{dom} \widetilde{\mathfrak{t}}$. Then $\mathfrak{L}$ satisfies the relation (5.5.1). Let $R_{\mathfrak{L}}$ be the operator given by (5.5.2). Then

$$\begin{aligned} \widetilde{\mathfrak{t}}[f, g] &= \mathfrak{t}_N[f, g] = \langle J^* \upharpoonright_{\mathfrak{L}} f, J^* \upharpoonright_{\mathfrak{L}} g\rangle \\ &= \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g\rangle = \langle R^{**}_{\mathfrak{L}} f, R^{**}_{\mathfrak{L}} g\rangle, \quad f, g \in \mathfrak{L}, \end{aligned}$$

and thus the linear relation $\widetilde{A}$ coincides with $R^*_{\mathfrak{L}} R^{**}_{\mathfrak{L}}$, cf. [40] and Proposition 4.5.2. ■

Corollary 5.5.2. *There is a one-to-one correspondence between the closed restrictions* $\widetilde{\mathfrak{t}}$ *of* $\mathfrak{t}_N$ *with* $\operatorname{dom} S \subset \operatorname{dom} \widetilde{\mathfrak{t}}$, *i.e.,*

$$\widetilde{\mathfrak{t}}[f, g] = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g\rangle, \quad f, g \in \mathfrak{L} = \operatorname{dom} \widetilde{\mathfrak{t}},$$

and the extremal nonnegative selfadjoint extensions $\widetilde{A}$ *of* S, *given by*

$$\widetilde{A} = R^*_{\mathfrak{L}} R_{\mathfrak{L}}, \quad \mathfrak{L} = \operatorname{dom} \widetilde{A}^{1/2}.$$

Proof. Let $\widetilde{\mathfrak{t}}$ be a closed restriction of the form $\mathfrak{t}_N$ with $\operatorname{dom} S \subset \operatorname{dom} \widetilde{\mathfrak{t}} =: \mathfrak{L}$. Then

$$\widetilde{\mathfrak{t}}[f,g] = \mathfrak{t}_N[f,g] = \langle J^* f, J^* g\rangle = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g\rangle, \quad f,\, g \in \mathfrak{L},$$

and the closedness of the form $\widetilde{\mathfrak{t}}$ implies that the operator $R_{\mathfrak{L}}$ is closed. Thus, $R_{\mathfrak{L}} = R_{\mathfrak{L}}^{**}$ and by Theorem 5.5.1 $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}$ is an extremal extension of S with $\mathfrak{L} = \operatorname{dom} \widetilde{A}^{1/2}$.

The mapping $\widetilde{\mathfrak{t}} \to \widetilde{A}$ is surjective, since if $\widetilde{A}$ is an extremal nonnegative selfadjoint extension of S, then with $\mathfrak{L} = \operatorname{dom} \widetilde{A}^{1/2}$ one has $\operatorname{dom} R_{\mathfrak{L}} = \mathfrak{L} = \operatorname{dom} \widetilde{\mathfrak{t}} = \operatorname{dom} R_{\mathfrak{L}}^{**}$ and $R_{\mathfrak{L}}^{**} = R_{\mathfrak{L}}$. Moreover, $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}$ and $\widetilde{\mathfrak{t}}[f,g] = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g\rangle$ is closed since $R_{\mathfrak{L}}$ is closed.

To see that the mapping is injective, let $\widetilde{\mathfrak{t}}_1$ and $\widetilde{\mathfrak{t}}_2$ be closed restrictions of $\mathfrak{t}_N$ such that $\operatorname{dom} S \subset \operatorname{dom} \widetilde{\mathfrak{t}}_i$, $i = 1, 2$, for which the corresponding selfadjoint extensions coincide, i.e.,

$$R_{\mathfrak{L}_1}^* R_{\mathfrak{L}_1} = R_{\mathfrak{L}_2}^* R_{\mathfrak{L}_2}.$$

This implies that $\operatorname{dom} \widetilde{\mathfrak{t}}_1 = \operatorname{dom} \widetilde{\mathfrak{t}}_2$ and since $\widetilde{\mathfrak{t}}_i \subset \mathfrak{t}_N$, $i = 1, 2$, the equality $\widetilde{\mathfrak{t}}_1 = \widetilde{\mathfrak{t}}_2$ follows. ∎

The one-to-one correspondence in Corollary 5.5.2 is between the extremal nonnegative selfadjoint extensions $\widetilde{A}$ of S and the closed restrictions $\widetilde{\mathfrak{t}}$ of the form $\mathfrak{t}_N$ to the subspaces $\mathfrak{L}\, (= \operatorname{dom} \widetilde{A}^{1/2})$ which satisfy

$$\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_N^{1/2}$$

or, equivalently,

$$\operatorname{dom} S_F^{1/2} \subset \mathfrak{L} \subset \operatorname{dom} S_N^{1/2}, \tag{5.5.20}$$

and which are closed subspaces in the form topology of $\mathfrak{t}_N$. If, for instance, $\mathfrak{L}$ is any subspace satisfying

$$\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_F^{1/2}$$

and $R_{\mathfrak{L}} = J^* \restriction \mathfrak{L}$, then $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = S_F$, since $R_{\mathfrak{L}}$ has the same closure as $Q = J^* \restriction \operatorname{dom} S$. In particular, the choice $\mathfrak{L} = \operatorname{dom} S_F^{1/2}$, $R_{\mathfrak{L}} = J^* \restriction \operatorname{dom} S_F^{1/2}$, gives the closure of the operator Q.

The next result reflects some further similar facts, cf. [10, Proposition 4.5].

Theorem 5.5.3. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$, let $\widetilde{A}$ be a nonnegative selfadjoint extension of S, and let $\mathfrak{L} := \operatorname{dom} \widetilde{A}$. Then:*

(i) $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} \leq \widetilde{A}$;

(ii) *if H is an extremal extension of S, such that*

$$R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} \leq H \leq \widetilde{A}, \tag{5.5.21}$$

then $H = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$;

(iii) $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = \widetilde{A}$ *if and only if* $\widetilde{A}$ *is an extremal extension of* S.

Proof. (i) Let $\widetilde{A}_{\mathfrak{L}} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$, let $\widetilde{\mathfrak{t}}_{\mathfrak{L}}$ be the closed form corresponding to the selfadjoint relation $\widetilde{A}_{\mathfrak{L}}$, and let $\widetilde{\mathfrak{t}}$ be the closed form corresponding to $\widetilde{A}$. Then $\operatorname{dom} \widetilde{A} = \mathfrak{L} \subset \operatorname{dom} \widetilde{\mathfrak{t}}_{\mathfrak{L}}$ and the inequality $S_N \leq \widetilde{A}$ leads to

$$\widetilde{\mathfrak{t}}_{\mathfrak{L}}[f,f] = \|R_{\mathfrak{L}}^{**} f\|_{\mathfrak{H}_S}^2 = \mathfrak{t}_N[f,f] \leq \widetilde{\mathfrak{t}}[f,f], \quad f \in \operatorname{dom} \widetilde{A}. \tag{5.5.22}$$

Therefore $\widetilde{A}_{\mathfrak{L}} \leq \widetilde{A}$ by Theorem 4.4.3.

(ii) Let H be an extremal extension of S such that (5.5.21) holds and let $\widetilde{\mathfrak{t}}_H$ be the closed form associated to H. Then $\operatorname{dom} \widetilde{A} \subset \operatorname{dom} \widetilde{\mathfrak{t}}_H$ and according to Theorem 5.5.1 one has $\widetilde{\mathfrak{t}}_H \subset \mathfrak{t}_N$. Therefore (5.5.22) gives

$$\widetilde{\mathfrak{t}}_{\mathfrak{L}}[f,g] = \widetilde{\mathfrak{t}}_H[f,g], \quad f, g \in \operatorname{dom} \widetilde{A}.$$

Since $\mathfrak{L} = \operatorname{dom} \widetilde{A}$ is a core for both $\widetilde{\mathfrak{t}}_{\mathfrak{L}}$ and $\widetilde{\mathfrak{t}}_H$, the equality $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = H$ follows.

(iii) Let $\widetilde{A}$ be an extremal extension of S. Then (i) and (ii) with $H = \widetilde{A}$ imply that $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = \widetilde{A}$. The converse statement is clear by Theorem 5.5.1. ∎

In the context of closed sectorial relations the equivalence (ii) ⇔ (iii) in Theorem 5.5.1 goes back to Arlinskiĭ [6]. The factorization of the extremal extensions of S in Theorem 5.5.1 leads to an explicit representation of the closed forms associated with the extremal extensions of S along the lines of the densely defined case in [10].

5.6 The case of bounded nonnegative operators

The constructions and the results in Section 5.3 and Section 5.5 can be made more explicit if the underlying nonnegative relation S is a bounded operator. In particular, this leads to an interpretation of the completion problem (5.1.1) as an extension problem as announced in the introduction.

5.6.1 Some general remarks

Let S be a closed bounded nonnegative operator in the Hilbert space $\mathfrak{H}$. Decompose the Hilbert space $\mathfrak{H}$ as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\operatorname{dom} S = \mathfrak{H}_1$. Then the operator S has the block decomposition

$$S = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}, \tag{5.6.1}$$

where $S_{11} \in [\mathfrak{H}_1]$ is nonnegative and $S_{21} \in [\mathfrak{H}_1, \mathfrak{H}_2]$. By Lemma 2.4.4 the adjoint of S in $\mathfrak{H}$ is the closed linear relation given by

$$S^* = \left\{ \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} S_{11}h_1 + S_{21}^* h_2 \\ \beta \end{pmatrix} \right\} : h_1 \in \mathfrak{H}_1, \, h_2, \beta \in \mathfrak{H}_2 \right\}, \tag{5.6.2}$$

and, in particular, it follows from (5.6.2) that

$$\operatorname{mul} S^* = \{0\} \oplus \mathfrak{H}_2. \tag{5.6.3}$$

The results from the earlier sections will be completely described in terms of the operators S_{11} and S_{21}. Since the operator S is nonnegative, it follows from Lemma 5.3.1 that the space $\mathfrak{R}_0$ in (5.3.3) is given by

$$\mathfrak{R}_0 = S(\ker S_{11}) = \{0\} \oplus \operatorname{ran}\,(S_{21} \restriction \ker S_{11})\,.$$

The linear operator Q from $\mathfrak{H}$ to $\mathfrak{H}_S$ in (5.3.6) is given by

$$Q = \left\{ \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right] \right\} \,:\, h_1 \in \mathfrak{H}_1 \right\}, \tag{5.6.4}$$

and the linear relation J from $\mathfrak{H}_S$ to $\mathfrak{H}$ in (5.3.7) is given by

$$J = \left\{ \left\{ \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right], \begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right\} \,:\, h_1 \in \mathfrak{H}_1 \right\}, \tag{5.6.5}$$

so that $\operatorname{mul} J = \mathfrak{R}_0 = \{0\} \oplus \operatorname{ran}\,(S_{21} \restriction \ker S_{11})$. Introduce the linear spaces

$$\mathfrak{M}_0 := \operatorname{ran} S_{11}^{1/2}, \quad \mathfrak{M} := \overline{\operatorname{ran}}\, S_{11}^{1/2} = \overline{\operatorname{ran}}\, S_{11},$$

so that $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}_1$ (with the original topology). Define the linear relation T_0 from $\mathfrak{M}$ to $\mathfrak{H}_S$ by

$$T_0 = \left\{ \left\{ S_{11}^{1/2}h_1, \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right] \right\} \,:\, h_1 \in \mathfrak{H}_1 \right\}, \tag{5.6.6}$$

so that

$$\left\| S_{11}^{1/2}h_1 \right\| = \left\| \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right] \right\|_{\mathfrak{H}_S}, \quad h_1 \in \mathfrak{H}_1,$$

where the norm in the righthand side is induced by the inner product (5.3.5). Thus, the relation T_0 is isometric from $\mathfrak{M}_0$ onto $\operatorname{ran} S/\mathfrak{R}_0$; and, in fact, T_0 is an operator. Hence, the closure T of T_0 is a closed isometric operator from the Hilbert space $\mathfrak{M}$ onto the Hilbert space $\mathfrak{H}_S$. Define the linear operator Q_1 from $\mathfrak{H}$ to $\mathfrak{M}$ by

$$Q_1 = \left\{ \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, S_{11}^{1/2}h_1 \right\} \,:\, h_1 \in \mathfrak{H}_1 \right\}, \tag{5.6.7}$$

so that Q_1 is not densely defined in $\mathfrak{H}$; in fact, $\operatorname{dom} Q_1 = \mathfrak{H}_1$. Define the linear relation J_1 from $\mathfrak{M}$ to $\mathfrak{H}$ by

$$J_1 = \left\{ \left\{ S_{11}^{1/2}h_1, \begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right\} \,:\, h_1 \in \mathfrak{H}_1 \right\}, \tag{5.6.8}$$

so that $\operatorname{mul} J_1 = \{0\} \oplus \operatorname{ran}\,(S_{21} \restriction \ker S_{11})$. Comparison of the definitions (5.6.4) and (5.6.7) shows that the linear operators Q_1 and Q are connected by

$$Q_1 = T^*Q, \tag{5.6.9}$$

and comparison of the definitions (5.6.5) and (5.6.8) shows that the linear relations J_1 and J are connected by

$$J_1 = JT. \tag{5.6.10}$$

The adjoints of Q and J have to be taken in terms of the Hilbert spaces $\mathfrak{H}_S$ and $\mathfrak{H}$. However, the relations Q_1 and J_1 are in the original Hilbert space and, hence, so are their adjoints.

Lemma 5.6.1. *The operator Q_1 in* (5.6.7) *is a closed operator, so that $Q_1^{**} = Q_1$. The adjoint Q_1^* of Q_1 is given by*

$$Q_1^* = \left\{ \left\{ g_1, \begin{pmatrix} S_{11}^{1/2} g_1 \\ g_2 \end{pmatrix} \right\} : \, g_1 \in \mathfrak{M}, \, g_2 \in \mathfrak{H}_2 \right\}, \tag{5.6.11}$$

and $\operatorname{mul} Q_1^* = \mathfrak{H}_2$.

Proof. Apply Lemma 2.4.4 to (2.4.4) with $\mathfrak{K}_1 = \mathfrak{H}_1$, $\mathfrak{K}_2 = \mathfrak{H}_2$, $\mathfrak{H}_1 = \mathfrak{M}$, $\mathfrak{Y} \subset \mathfrak{H}_2 = \emptyset$, $D = \{0, 0\}$, and $B = S_{11}^{1/2}$. Then clearly $D^* = \mathfrak{M} \times \mathfrak{H}_2$ and (2.4.7) leads to (5.6.11). ∎

Define the linear relation W from $\mathfrak{M}$ to $\mathfrak{H}_2$ as follows

$$W = \{ \{ S_{11}^{1/2} g, S_{21} g \} : \, g \in \mathfrak{H}_1 \},$$

so that W is densely defined and $\{0\} \oplus \operatorname{mul} W = \mathfrak{R}_0$. The adjoint $V = W^*$ is a closed operator from $\mathfrak{H}_2$ to $\mathfrak{M}$ given by

$$V = \{ \{h, k\} : \, S_{11}^{1/2} k = S_{21}^* h \}. \tag{5.6.12}$$

The definition of $\mathfrak{M}$ shows directly that

$$\operatorname{dom} V = \{ h : \, S_{21}^* h \in \operatorname{ran} S_{11}^{1/2} \}, \tag{5.6.13}$$

and $S_{11}^{1/2} V h = S_{21}^* h$, $h \in \operatorname{dom} V$. In fact, the operator V is given by

$$V = S_{11}^{(-1/2)} S_{21}^*,$$

where $S_{11}^{(-1/2)}$ is the *pseudo-inverse* of $S_{11}^{1/2}$ defined as the operator that assigns to an element in $\operatorname{ran} S_{11}^{1/2}$ its uniquely defined original in $\mathfrak{H}_1 \ominus \ker S_{11} = \overline{\operatorname{ran}}\, S_{11} = \mathfrak{M}$. Observe that

$$S_{11}^{1/2} V = S_{21}^* \restriction \operatorname{dom} V. \tag{5.6.14}$$

The identity $V^* = W^{**}$ shows that W is closable if and only if V is densely defined. It is clear from (5.6.8) that J_1 can be rewritten in the form

$$J_1 = \left\{ \left\{ h_1, \begin{pmatrix} S_{11}^{1/2} h_1 \\ k_1 \end{pmatrix} \right\} : \, \{h_1, k_1\} \in W \right\}. \tag{5.6.15}$$

Lemma 5.6.2. *Let the relation* J_1 *from* $\mathfrak{M}$ *to* $\mathfrak{H}$ *be defined by* (5.6.8). *Then* J_1^* *is an operator from* $\mathfrak{H}$ *to* $\mathfrak{M}$ *given by*

$$J_1^* = \left\{ \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, S_{11}^{1/2} g_1 + V g_2, \right\} \,:\, g_1 \in \mathfrak{H}_1,\, g_2 \in \operatorname{dom} V \right\}. \tag{5.6.16}$$

Its adjoint relation J_1^{**} *from* $\mathfrak{M}$ *to* $\mathfrak{H}$ *is given by*

$$J_1^{**} = \left\{ \left\{ \alpha, \begin{pmatrix} S_{11}^{1/2} \alpha \\ \beta \end{pmatrix} \right\} \,:\, \{\alpha, \beta\} \in V^* \right\}, \tag{5.6.17}$$

and $\operatorname{mul} J_1^{**} = \operatorname{mul} V^*$.

Proof. Apply Lemma 2.4.4 with $\mathfrak{X} \subset \mathfrak{H}_2 = \emptyset$ and $\mathfrak{H} = \mathfrak{H}_1$ to obtain (5.6.16) from (5.6.15). In fact, (5.6.17) is a direct consequence of (5.6.15). The description of $\operatorname{mul} J_1^{**}$ is a consequence of (5.6.17), see also (2.4.10). ■

With the above identification of the spaces $\mathfrak{H}_S$ and $\mathfrak{M}$ under the isometry T, it is possible to translate the results involving Q and J in terms of Q_1 and J_1. It will be helpful to use the following notations

$$Q_1 = \begin{pmatrix} S_{11}^{1/2} & O \end{pmatrix}, \quad Q_1^* = \begin{pmatrix} S_{11}^{1/2} \\ O^* \end{pmatrix},$$

where O stands for the trivial linear relation from $\mathfrak{H}_2$ to $\mathfrak{M}$, so that $O^* = \mathfrak{M} \times \mathfrak{H}_2$, and

$$J_1^* = \begin{pmatrix} S_{11}^{1/2} & V \end{pmatrix}, \quad J_1^{**} = \begin{pmatrix} S_{11}^{1/2} \\ V^* \end{pmatrix}.$$

5.6.2 The Kreĭn-von Neumann and the Friedrichs extensions

The following result is a straightforward translation of Theorem 5.3.3.

Theorem 5.6.3. *Let* S *in* (5.6.1) *be a closed bounded nonnegative operator in* $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ *with* $\operatorname{dom} S = \mathfrak{H}_1$. *Then the Kreĭn-von Neumann extension* S_N *of* S *is given by*

$$S_N = \begin{pmatrix} S_{11}^{1/2} \\ V^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & V \end{pmatrix}, \tag{5.6.18}$$

and the corresponding closed form $\mathfrak{t}_N$ *is given by*

$$\mathfrak{t}_N \left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right] = \left\| S_{11}^{1/2} h_1 + V h_2 \right\|^2, \quad h_1 \in \mathfrak{H}_1,\, h_2 \in \operatorname{dom} V. \tag{5.6.19}$$

Furthermore, the Friedrichs extension S_F *of* S *is given by*

$$S_F = \begin{pmatrix} S_{11}^{1/2} \\ O^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & O \end{pmatrix}, \tag{5.6.20}$$

and the corresponding closed form $\mathfrak{t}_F$ *is given by*

$$\mathfrak{t}_F\left[\begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] = \left\| S_{11}^{1/2} h_1 \right\|^2, \quad h_1 \in \mathfrak{H}_1. \tag{5.6.21}$$

Proof. First consider the Kreĭn-von Neumann extension. Lemma 2.4.7 and (5.6.8) lead to

$$J_1^* = T^* J^*, \quad J_1^{**} = J^{**} T. \tag{5.6.22}$$

By multiplying the relations in (5.6.22) it follows that

$$J_1^{**} J_1^* = J^{**} T T^* J^* = J^{**} J^*,$$

and hence $J_1^{**} J_1^* = J^{**} J^* = S_N$ by Theorem 5.3.3. Hence (5.6.19) is clear, see Proposition 4.5.2.

Now consider the Friedrichs extension. Lemma 2.4.7 and (5.6.7) lead to

$$Q_1^* = Q^* T, \quad Q_1^{**} = T^* Q^{**}. \tag{5.6.23}$$

By multiplying the relations in (5.6.23) one obtains

$$Q_1^* Q_1^{**} = Q^* T T^* Q^{**} = Q^* Q^{**},$$

and hence $Q_1^* Q_1^{**} = Q^* Q^{**} = S_F$ by Theorem 5.3.3. Hence (5.6.21) is clear, see Proposition 4.5.2. ∎

The factorizations in (5.6.18) and (5.6.20) may be rewritten more explicitly. It follows from (5.6.18) that

$$\begin{aligned} S_N \;=\; & \left\{ \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} S_{11} h_1 + S_{11}^{1/2} V h_2 \\ \beta \end{pmatrix} \right\} : \right. \\ & \left. h_1 \in \mathfrak{H}_1,\, h_2 \in \operatorname{dom} V,\, \{ S_{11}^{1/2} h_1 + V h_2, \beta \} \in V^* \right\}. \end{aligned} \tag{5.6.24}$$

Observe that according to (5.6.14) one has $S_{11}^{1/2} V \subset S_{21}^*$, which shows that the righthand side of (5.6.24) is indeed a restriction of the righthand side of (5.6.2). Furthermore, note that it also follows from (5.6.14) that $\{ S_{11}^{1/2} h_1, S_{21} h_1 \} \in V^*$. Hence S_N may be also written as

$$S_N = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & V^* V \end{pmatrix}.$$

It follows from (5.6.20) that

$$S_F = \left\{ \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} S_{11} h_1 \\ \varphi \end{pmatrix} \right\} : h_1 \in \mathfrak{H}_1,\, \varphi \in \mathfrak{H}_2 \right\}. \tag{5.6.25}$$

As a consequence of (5.6.18) one obtains that

$$\operatorname{mul} S_N = \{0\} \oplus \operatorname{mul} V^*.$$

Hence the following corollary is straightforward.

Corollary 5.6.4. *Let S in* (5.6.1) *be a bounded nonnegative operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with* $\operatorname{dom} S = \mathfrak{H}_1$. *Then the following conditions are equivalent:*

(i) S_N *is an operator;*

(ii) W *is a closable operator from $\mathfrak{M}$ to $\mathfrak{H}_2$;*

(iii) V *is a densely defined operator from $\mathfrak{H}_2$ to $\mathfrak{M}$.*

Furthermore, the following conditions are equivalent:

(iv) S_N *belongs to* $[\mathfrak{H}]$*;*

(v) $W \in [\mathfrak{M}, \mathfrak{H}_2]$*;*

(vi) $V \in [\mathfrak{H}_2, \mathfrak{M}]$.

The conditions (i), (ii), and (iii) are equivalent to S_N and S_F being disjoint, cf. Proposition 5.4.11. The conditions (iv), (v), and (vi) are equivalent to S_N and S_F being transversal, cf. Proposition 5.4.13.

Corollary 5.6.5. *Let S in* (5.6.1) *be a bounded nonnegative operator in $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $S_F = S_N$, *i.e. S has a unique nonnegative selfadjoint extension;*

(ii) $\operatorname{dom} V = \{0\}$*;*

(iii) $\operatorname{ran} S_{11}^{1/2} \cap \operatorname{ran} S_{21}^* = \{0\}$ *and* $\ker S_{21}^* = \{0\}$.

Proof. (i) $\Leftrightarrow$ (ii) Comparing (5.6.19) and (5.6.21) in Theorem 5.6.3 it is clear that $S_N = S_F$ if and only if $\operatorname{dom} V = \{0\}$.

(ii) $\Leftrightarrow$ (iii) This follows from (5.6.13). ■

The equivalence of the items (i), (ii), and (iii) in Corollary 5.6.4 can be found in [37] under the technical condition that $\ker S_{11} = \{0\}$. Items (iv), (v), and (vi) in Corollary 5.6.4 go back to [2]. The condition (vi) is equivalent to the condition $\operatorname{ran} S_{21}^* \subset \operatorname{ran} S_{11}^{1/2}$.

5.6.3 Extremal extensions

In order to consider the extremal extensions of S, let $\mathfrak{L}_2$ be any (not necessarily closed) subspace of $\mathfrak{H}_2$ such that $\mathfrak{L}_2 \subset \operatorname{dom} V$, let $R_{\mathfrak{L}}$ be the operator from $\mathfrak{H}$ to $\mathfrak{H}_S$ given by

$$R_{\mathfrak{L}} = J^* \upharpoonright \mathfrak{H}_1 \oplus \mathfrak{L}_2, \tag{5.6.26}$$

and let $R_{1\mathfrak{L}}$ be the operator from $\mathfrak{H}$ to $\mathfrak{M}$, defined by

$$R_{1\mathfrak{L}} = \left\{ \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, S_{11}^{1/2} g_1 + V g_2, \right\} \, : \, g_1 \in \mathfrak{H}_1, \, g_2 \in \mathfrak{L}_2 \right\}. \tag{5.6.27}$$

Comparison of the definitions (5.6.26) and (5.6.27) shows that the linear operators $R_{1\mathfrak{L}}$ and $R_{\mathfrak{L}}$ are connected by $R_{1\mathfrak{L}} = T^* R_{\mathfrak{L}}$, so that

$$R_{1\mathfrak{L}}^* R_{1\mathfrak{L}}^{**} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}. \tag{5.6.28}$$

The following notation turns out to be useful: the restriction $V \upharpoonright \mathfrak{L}_2$ will be denoted by $V_{\mathfrak{L}_2}$.

Lemma 5.6.6. *Let the operator $R_{1\mathfrak{L}}$ from $\mathfrak{H}$ to $\mathfrak{M}$ be defined by* (5.6.27). *Then the relation $R_{1\mathfrak{L}}^*$ from $\mathfrak{M}$ to $\mathfrak{H}$ is given by*

$$R_{1\mathfrak{L}}^* = \left\{ \left\{ \alpha, \begin{pmatrix} S_{11}^{1/2}\alpha \\ \beta \end{pmatrix} \right\} : \{\alpha, \beta\} \in V_{\mathfrak{L}_2}^* \right\}, \tag{5.6.29}$$

and the operator $R_{1\mathfrak{L}}^{**}$ from $\mathfrak{H}$ to $\mathfrak{M}$ is given by

$$R_{1\mathfrak{L}}^{**} = \left\{ \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, S_{11}^{1/2} g_1 + V_{\mathfrak{L}_2}^{**} g_2, \right\} : g_1 \in \mathfrak{H}_1,\, g_2 \in \operatorname{dom} V_{\mathfrak{L}_2}^{**} \right\}. \tag{5.6.30}$$

Proof. Again the statements are obtained from Lemma 2.4.4. Observe that $V_{\mathfrak{L}_2}^{**}$ is an operator as a restriction of the closed operator V. The identity (5.6.30) follows also directly from (5.6.27). ∎

Now, taking into account (5.6.28) and Lemma 5.6.6, a characterization of all nonnegative extremal extensions of S can be easily obtained via Theorem 5.5.1. It will be helpful to use the following notation:

$$R_{1\mathfrak{L}}^* = \begin{pmatrix} S_{11}^{1/2} \\ V_{\mathfrak{L}_2}^* \end{pmatrix}, \quad R_{1\mathfrak{L}}^{**} = \begin{pmatrix} S_{11}^{1/2} & V_{\mathfrak{L}_2}^{**} \end{pmatrix}.$$

Proposition 5.6.7. *Let S in* (5.6.1) *be a bounded nonnegative operator in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\operatorname{dom} S = \mathfrak{H}_1$ and let $\widetilde{A}$ be a nonnegative selfadjoint extension of S. Then the following statements are equivalent:*

(i) *$\widetilde{A}$ is an extremal extension of S;*

(ii) *$\widetilde{A}$ is is of the form*

$$\widetilde{A} = \begin{pmatrix} S_{11}^{1/2} \\ V_{\mathfrak{L}_2}^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & V_{\mathfrak{L}_2}^{**} \end{pmatrix},$$

with a subspace $\mathfrak{L}_2$ of $\mathfrak{H}_2$, $\mathfrak{L}_2 \subset \operatorname{dom} V$;

(iii) *the closed form $\widetilde{\mathfrak{t}}$ associated to $\widetilde{A}$ is given by*

$$\widetilde{\mathfrak{t}}\left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right] = \left\| S_{11}^{1/2} h_1 + V_{\mathfrak{L}_2}^{**} h_2 \right\|^2, \quad h_1 \in \mathfrak{H}_1, \quad h_2 \in \operatorname{dom} V_{\mathfrak{L}_2}^{**}, \tag{5.6.31}$$

with a subspace $\mathfrak{L}_2$ of $\mathfrak{H}_2$, $\mathfrak{L}_2 \subset \operatorname{dom} V$.

Furthermore, (5.6.31) *establishes a one-to-one correspondence between the extremal extensions* $\widetilde{A}$ *of* S *and the closed restrictions* $V_{\mathfrak{L}_2}$ *of* V*, or equivalently, those subspaces* $\mathfrak{L}_2 \subset \operatorname{dom} V$ *for which the restriction* $V_{\mathfrak{L}_2}$ *is a closed operator.*

Proof. The equivalence of the statements (i)–(iii) follows immediately from Theorem 5.5.1 and Lemma 5.6.6. As to the last statement observe, that $\widetilde{\mathfrak{t}}$ in (5.6.31) is closed if and only if the block operator $\begin{pmatrix} S_{11}^{1/2} & V_{\mathfrak{L}_2} \end{pmatrix}$ is closed. Since here $S_{11}^{1/2}$ is a closed bounded operator, this block operator is closed precisely when $V_{\mathfrak{L}_2}$ is closed. ■

Again, the factorization in Proposition 5.6.7 may be rewritten explicitly. The form of S^* in (5.6.2) leads to the following representation:

$$\begin{aligned} \widetilde{A} \;=\; & \left\{ \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} S_{11}h_1 + S_{11}^{1/2} V_{\mathfrak{L}_2}^{**} h_2 \\ \beta \end{pmatrix} \right\} : \right. \\ & \left. \qquad h_1 \in \mathfrak{H}_1,\ h_2 \in \operatorname{dom} V_{\mathfrak{L}_2}^{**},\ \{S_{11}^{1/2}h_1 + V_{\mathfrak{L}_2}^{**}h_2, \beta\} \in V_{\mathfrak{L}_2}^{*} \right\}. \end{aligned}$$

It is an immediate consequence of (5.6.14) that $\{S_{11}^{1/2}h_1, S_{21}h_1\} \in V_{\mathfrak{L}_2}^*$. Therefore also $\{V_{\mathfrak{L}_2}^{**}h_2, \beta - S_{21}h_1\} \in V_{\mathfrak{L}_2}^*$. Hence $\widetilde{A}$ may also be rewritten as

$$\widetilde{A} = \begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & V_{\mathfrak{L}_2}^* V_{\mathfrak{L}_2}^{**} \end{pmatrix}. \tag{5.6.32}$$

Note that $\operatorname{mul} V_{\mathfrak{L}_2}^* V_{\mathfrak{L}_2}^{**} = \operatorname{mul} V_{\mathfrak{L}_2}^* = \mathfrak{H}_2 \ominus (\operatorname{clos} \mathfrak{L}_2)$.

Remark 5.6.8. *Clearly, Proposition 5.6.7 implies that an extremal extension* $\widetilde{A}$ *of the bounded nonnegative operator* S *is an operator if and only if*

$$\overline{\operatorname{dom}}\, V_{\mathfrak{L}_2} = \overline{\operatorname{dom}}\, V_{\mathfrak{L}_2}^{**} = \mathfrak{H}_2.$$

Now assume that $\widetilde{A} \in [\mathfrak{H}]$ *is a nonnegative extremal extension of* S*. Then, in particular,* $S_N \in [\mathfrak{H}]$ *by Proposition 5.4.12. Furthermore,* $\operatorname{dom} V_{\mathfrak{L}_2}^{**} = \mathfrak{H}_2$ *and, since* $V_{\mathfrak{L}_2}^{**}$ *is a closed restriction of the closed operator* V*, the equality* $V_{\mathfrak{L}_2}^{**} = V$ *follows, so that* $\widetilde{A} = S_N$*. Hence, if* $S_N \in [\mathfrak{H}]$*, then* S_N *is the only nonnegative extremal selfadjoint extension of* S *which belongs to* $[\mathfrak{H}]$*.*

5.7 Special symmetric relations

The extremal extensions of a nonnegative relation in a Hilbert spaces have been characterized in Theorem 5.5.1. In this section this characterization will be made explicit for some simple nonnegative relations.

5.7.1 Null operators

Let $\mathfrak{K}$ be a not necessarily closed subspace of the Hilbert space $\mathfrak{H}$ and define the null operator S on $\mathfrak{K}$ by

$$S = \mathfrak{K} \times \{0\}. \tag{5.7.1}$$

Then S is (the graph of) a nonnegative bounded operator, which is closed if and only if $\mathfrak{K}$ is closed. The adjoint S^* is a relation, given by

$$S^* = \mathfrak{H} \times \mathfrak{K}^\perp, \tag{5.7.2}$$

so that $\ker\,(S^* + a) = \mathfrak{K}^\perp$, $a \in \mathbb{R}$, and $\operatorname{mul} S^* = \mathfrak{K}^\perp$. Since S is nonnegative, the construction in Section 5.3 is applicable and

$$\mathfrak{R}_0 = \{0\}, \quad \mathfrak{H}_S = \{0\}, \quad Q = \mathfrak{K} \times \{0\}, \quad J = \{0\} \times \{0\},$$

so that

$$Q^* = \{0\} \times \mathfrak{K}^\perp, \quad Q^{**} = \overline{\mathfrak{K}} \times \{0\}, \quad J^* = \mathfrak{H} \times \{0\}, \quad J^{**} = \{\{0,0\}\},$$

where $\overline{\mathfrak{K}}$ denotes the closure of $\mathfrak{K}$. Therefore, according to Theorem 5.3.3, the Kreĭn-von Neumann extension $S_N = J^{**}J^*$ and the Friedrichs extension $S_F = Q^*Q^{**}$ are given by

$$S_N = \mathfrak{H} \times \{0\}, \quad S_F = \overline{\mathfrak{K}} \times \mathfrak{K}^\perp. \tag{5.7.3}$$

Note that (5.7.1) and (5.7.2) imply that $S_N = \operatorname{clos} S \,\widehat{+}\, (\ker\, S^* \times \{0\})$. Also note that (5.7.1) and (5.7.2) imply that $S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*) = \mathfrak{K} \times \mathfrak{K}^\perp$. Therefore the equality $S_F = S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*)$, which is equivalent to $\operatorname{dom} S = \overline{\operatorname{dom} S \cap \operatorname{dom} S^*}$, is satisfied if and only if $\mathfrak{K}$ is closed, cf. Section 5.2.

Clearly, (5.7.3) shows that S_N is the null operator defined on all of $\mathfrak{H}$, and this is equivalent to the transversality of S_N and S_F: $S^* = S_N \,\widehat{+}\, S_F$, cf. Proposition 5.4.13. Moreover, $S_N \cap S_F = \overline{\mathfrak{K}} \times \{0\} = \operatorname{clos} S$, so that S_N and S_F are disjoint with respect to $\operatorname{clos} S$, and they are disjoint (with respect to S) precisely when $\mathfrak{K} = \overline{\mathfrak{K}}$, cf. Proposition 5.4.11. Furthermore, observe that

$$\ker\,(S^* + a) \cap \operatorname{dom} S_N^{1/2} = \mathfrak{K}^\perp, \quad a > 0.$$

Hence, $S_N = S_F$ if and only if $\mathfrak{K}$ is dense in $\mathfrak{H}$, which may also be verified directly (cf. Theorem 5.4.7).

The form corresponding to S_N is the null form defined on all of $\mathfrak{H}$, while the form corresponding to S_F is the null form on $\overline{\mathfrak{K}}$.

Lemma 5.7.1. *Let the operator S be given by* (5.7.1)*. Then there exists a one-to-one correspondence between the set of all extremal extensions $\widetilde{A}$ of S and the set of all closed subspaces $\mathfrak{L}$ with $\overline{\mathfrak{K}} \subset \mathfrak{L} \subset \mathfrak{H}$. This correspondence is given by*

$$\widetilde{A} = \mathfrak{L} \times \mathfrak{L}^\perp, \tag{5.7.4}$$

and the associated form $\widetilde{\mathfrak{t}}$ is the null form on $\mathfrak{L}$.

Proof. Apply Theorem 5.5.1 with S given by (5.7.1). Note that $\operatorname{dom} S = \mathfrak{K}$ and $\operatorname{dom} S_N = \operatorname{dom} S_N^{1/2} = \mathfrak{H}$, by (5.7.1) and (5.7.3). Then the inclusions in (5.5.1) correspond to

$$\mathfrak{K} \subset \mathfrak{L} \subset \mathfrak{H},$$

and, since $J^* = \mathfrak{H} \times \{0\}$, it follows that $R_{\mathfrak{L}}$ in (5.5.2) is given by

$$R_{\mathfrak{L}} = \mathfrak{L} \times \{0\},$$

so that

$$R_{\mathfrak{L}}^* = \{0\} \times \mathfrak{L}^\perp, \quad R_{\mathfrak{L}}^{**} = \overline{\mathfrak{L}} \times \{0\}.$$

Therefore, $\widetilde{A}$ in Theorem 5.5.1 is given by (5.7.4). The statement about the corresponding form is obvious. ■

The choice $\mathfrak{K} = \{0\}$ provides an interesting case since then $S = \{\{0,0\}\}$ and $S^* = \mathfrak{H} \times \mathfrak{H}$. Furthermore by (5.7.3), $S_N = \mathfrak{H} \times \{0\}$ and $S_F = \{0\} \times \mathfrak{H}$. The set of all extremal extensions $\widetilde{A}$ of $S = \{\{0,0\}\}$ and the set of all closed subspaces $\mathfrak{L}$ of $\mathfrak{H}$ are in one-to-one correspondence via (5.7.4). The associated form $\widetilde{\mathfrak{t}}$ is the null form on $\mathfrak{L}$.

5.7.2 Purely multivalued relations

Let $\mathfrak{K}$ be a not necessarily closed subspace of the Hilbert space $\mathfrak{H}$ and define the purely multivalued relation S by

$$S = \{0\} \times \mathfrak{K}. \tag{5.7.5}$$

Then S is closed if and only if $\mathfrak{K}$ is closed. The adjoint S^* is given by

$$S^* = \mathfrak{K}^\perp \times \mathfrak{H}, \tag{5.7.6}$$

so that $\operatorname{mul} S^* = \mathfrak{H}$ and $\ker\,(S^* + a) = \mathfrak{K}^\perp$, $a \in \mathbb{R}$. Since S is nonnegative, the construction in Section 5.2 is applicable and

$$\mathfrak{R}_0 = \operatorname{ran} S = \mathfrak{K}, \quad \mathfrak{H}_S = \{[0]\}, \quad Q = \{\{0,[0]\}\}, \quad J = \{[0]\} \times \mathfrak{K},$$

so that

$$Q^* = \{[0]\} \times \mathfrak{H}, \quad Q^{**} = \{0\} \times \{[0]\}, \quad J^* = \mathfrak{K}^\perp \times \{[0]\}, \quad J^{**} = \{[0]\} \times \overline{\mathfrak{K}}.$$

Therefore, according to Theorem 5.3.3, the Kreĭn-von Neumann extension $S_N = J^{**}J^*$ and the Friedrichs extension $S_F = Q^*Q^{**}$ are given by

$$S_N = \mathfrak{K}^\perp \times \overline{\mathfrak{K}}, \quad S_F = \{0\} \times \mathfrak{H}. \tag{5.7.7}$$

Note that (5.7.5) and (5.7.6) imply that $S \,\widehat{+}\, (\ker S^* \times \{0\}) = \mathfrak{K}^\perp \times \mathfrak{K}$. Therefore the equality $S_N = S \,\widehat{+}\, (\ker S^* \times \{0\})$, which is equivalent to $\operatorname{ran} S = \overline{\operatorname{ran}}\, S \cap \operatorname{ran} S^*$, is satisfied if and only if $\mathfrak{K}$ is closed. Also note that (5.7.5) and (5.7.6) imply that $S_F = \operatorname{clos} S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*)$.

Clearly, (5.7.7) shows that S_N and S_F are transversal. Moreover, $S_N \cap S_F = \{0\} \times \overline{\mathfrak{K}}$, so that S_N and S_F are disjoint with respect to $\operatorname{clos} S$, and they are disjoint (with respect to S) precisely when $\mathfrak{K}$ is closed. Furthermore, observe that

$$\ker\,(S^* + a) \cap \operatorname{dom} S_N^{1/2} = \mathfrak{K}^\perp, \quad a > 0.$$

Hence, $S_N = S_F$ if and only if $\mathfrak{K}$ is dense in $\mathfrak{H}$, which may also be verified directly (cf. Theorem 5.4.7).

The form corresponding to S_N is the null form defined on all of $\mathfrak{K}^\perp$, while the form corresponding to S_F is the null form on $\{0\}$.

Lemma 5.7.2. *Let the relation S be given by* (5.7.5)*. Then there exists a one to-one-correspondence between the set of all extremal extensions $\widetilde{A}$ of S and the set of all closed subspaces $\mathfrak{L}$ of $\mathfrak{K}^\perp$. The correspondence is given by*

$$\widetilde{A} = \mathfrak{L} \times \mathfrak{L}^\perp, \tag{5.7.8}$$

and the associated form $\widetilde{\mathfrak{t}}$ is the null form on $\mathfrak{L}$.

Proof. Apply Theorem 5.5.1 with S given by (5.7.5). Note that $\operatorname{dom} S = \{0\}$ and $\operatorname{dom} S_N = \operatorname{dom} S_N^{1/2} = \mathfrak{K}^\perp$, by (5.7.5) and (5.7.7). Then the inclusions in (5.5.1) correspond to

$$\mathfrak{L} \subset \mathfrak{K}^\perp,$$

and, since $J^* = \mathfrak{K}^\perp \times \{0\}$, it follows that $R_\mathfrak{L}$ in (5.5.2) is given by

$$R_\mathfrak{L} = \mathfrak{L} \times \{[0]\},$$

so that

$$R_\mathfrak{L}^* = \{[0]\} \times \mathfrak{L}^\perp, \quad R_\mathfrak{L}^{**} = \overline{\mathfrak{L}} \times \{[0]\}.$$

Therefore, $\widetilde{A}$ in Theorem 5.5.1 is given by (5.7.8). The statement about the corresponding form is obvious. ∎

The choice $\mathfrak{K} = \{0\}$ leads again to the case $S = \{\{0,0\}\}$ and $S^* = \mathfrak{H} \times \mathfrak{H}$. Furthermore by (5.7.7), $S_N = \mathfrak{H} \times \{0\}$ and $S_F = \{0\} \times \mathfrak{H}$. The set of all extremal extensions $\widetilde{A}$ of $S = \{\{0,0\}\}$ and the set of all closed subspaces $\mathfrak{L}$ of $\mathfrak{H}$ are in one-to-one correspondence via (5.7.4). The associated form $\widetilde{\mathfrak{t}}$ is the null form on $\mathfrak{L}$.

Note that Lemma 5.7.1 and Lemma 5.7.2 are intimately connected. This is best seen via the identity

$$S_N = ((S^{-1})_F)^{-1},$$

which holds for any nonnegative relation S, cf. [2], [20].

Conclusions

In this chapter certain factorizations of nonnegative selfadjoint relations have been used in order to study the extremal extensions of a nonnegative relation in a Hilbert space. The disjointness and transversality for nonnegative selfadjoint extensions of a nonnegative relation have been characterized. Factorizations of the Kreĭn-von Neumann and the Friedrichs extensions of a nonnegative relation have been constructed. Disjointness, transversality and equality of the Kreĭn-von Neumann and Friedrichs extensions have been discussed. The question whether the Kreĭn-von Neumann extension is a (bounded) operator has been also discussed. All extremal extensions of a nonnegative relation have been characterized in terms of their factorizations. The case of a nonnegative bounded operator and its extremal extensions has been also treated.

There are at least two open problems in connection with the extension theory of nonnegative relations in Hilbert spaces:

1. Characterize all nonnegative selfadjoint extensions of a nonnegative relation using the factorization approach proposed in this chapter.

2. Give a factorization of the extremal extensions in exit spaces of a nonnegative relation in a Hilbert space. Use the factorization to characterize all nonnegative selfadjoint extensions in exist spaces of a nonnegative relation.